A DRILL-BOOK IN TRIGONOMETRY

BY

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PREFACE.

In 1881 a TREATISE ON TRIGONOMETRY was published under the joint authority of PROFESSORS OLIVER, WAIT, and JONES. In 1889 this book was rewritten and reissued under the same title and by the same authority. In all five editions have been printed.

Professor Oliver died last March, and now that a new edition of the book is called for and many changes are proposed, it seems better, perhaps fairer towards him, to issue it under my single name. It may be regarded, then, both as a new edition of the older book, and as itself a new book.

Among the more important changes are these:

1. The introduction, at the beginning, of a chapter on THE RIGHT TRIANGLE, treating it as the pupil has been accustomed to think of it in plane geometry, and without the complex notions of directed lines and angles.

In this chapter he learns, also, how to use tables of trigonometric ratios and logarithms, and he gets some notion of the simpler applications of trigonometry to problems in surveying.

2. The second chapter, on the GENERAL PROPERTIES OF PLANE ANGLES, follows more closely the general lines of the old treatise, but it differs widely in details: in particular, it makes a much freer use of projections.

3. The third chapter, on PLANE TRIANGLES, shows a more radical departure. The habit of writers on trigonometry seems to have been to give broad and general definitions of trigonometric ratios, and to prove generally the propositions that relate to plane angles, and then, when they come to discuss the properties of plane triangles, to forget all they had said before, and to fall back on the ratios of positive acute angles.

In the edition of 1889 I tried to make the definitions and the proofs general; but the method then followed never satis-

fied me, and I sought in vain for light in the many American and foreign text-books that I consulted.

But now, through a happy suggestion of one of my assistants, Mr. Fowler, I think I have overcome the difficulty. That suggestion was to use THE EXTERIOR ANGLES; and by such use I have been able to make the proofs general and the formulæ symmetric. So, in space trigonometry, I have been able to apply this suggestion to the discussion of the properties of triedral angles and spherical triangles with the best results.

4. Greater prominence has been given to the GENERAL TRIANGLES.

5. The proof of De Moivre's formula by aid of imaginaries has been left out: I propose to write a book, shortly, on HIGHER ALGEBRA, and it has seemed to me that there would be the best place to discuss the applications of imaginaries to trigonometry.

6. Most of the FIGURES have been redrawn.

On the other hand, many parts of the older book have been included without change, notably the discussion of derivatives and series, of directed areas, of astronomy, and of navigation; and for the most part the examples have been taken bodily.

As to the title of the book, it has seemed to me that the word TREATISE was too large for me; and as I have meant my book primarily for class use, I have called it a DRILL-BOOK.

In writing this book, I have been very fortunate in my assistants. To Mr. Charles S. Fowler and Dr. Virgil Snyder, instructors in mathematics in Cornell University, I am deeply indebted, both for their valuable suggestions, and for their unwearied labors in beating out the text and in preparing the questions and examples; and, for its dress, I am no less indebted to my draughtsmen, Mr. John S. Reid and Mr. Hiram S. Gutsell, instructors in drawing, to my engravers, the American Bank Note Company, and to my printers, Messrs. J. J. Little & Co.

GEORGE W. JONES.

ITHACA, N. Y., *January* 1, 1896.

SUGGESTIONS TO TEACHERS.

THERE are many things in this book not meant for beginners. Below is a rough list of the chapters and parts of chapters that may be taken up at a first reading: the parts omitted are for advanced classes. And as to those parts which are included in the list, great caution must be taken lest too many examples, or too hard ones, be set; for there are many of them, printed in a small space. No one can be expected to work them all, and the hardest of them should be reserved for the strongest pupils. But the profit comes to the pupil by hard thinking; and the best part of the thinking is in answering the questions.

Very often more than one figure is used to illustrate a principle: for the most part, the first figure is the simplest, and that one should be well understood before the others are looked at. Later the other figures may be taken up, and the generality of the principle will be felt only when they have all been studied.

When the reasons are obvious, both theorems and corollaries are left without formal demonstration; but students are expected to state the proofs.

In most cases theorems are given only in formula: it is best that these formulæ be translated into words.

In most cases answers to the examples are not given, and the student is left to test his own results: the testing is counted as not less important than the solution, and the habit of independent thought and self-reliance so cultivated as most valuable of all.

Only the main lines of the subject are developed in the text: collateral matters are outlined in the examples and left for the student to work out for himself.

FOR A FIRST READING.

I,	all,	pp. 1–21.
II,	§§ 1–9, 12,	pp. 22–53, 58–60.
III,	§§ 1–4,	pp. 62–75.
IV,	none.	
V,	§§ 1–7, 9–15,	pp. 104–130, 134–161.

NEW SIGNS AND WORDS.

SOME of the less familiar signs used in this book are these :

$\geqslant$, *larger than* ; $\ngeqslant$, *not larger than* ;

$\leqslant$, *smaller than* ; $\nleqslant$, *not smaller than* ;

$\ngtr$, *not greater than* ; $\nless$, *not less than* ;

$\neq$, *not equal to* ; $\cdots$, *and so on*, meaning the continuance of a series of terms in the way it has begun ;

$\doteq$, *approaches*, meaning that the value of one expression comes very close to that of another, without absolute equality ;

$\equiv$, *stands for*, or *is identical with*.

The common point of two or more lines or planes is their *co-point* ; the common line of two or more points or planes is their *co-line* ; and the common plane of two or more points or lines is their *co-plane*. The corresponding adjectives are *co-pointar*, *co-linear*, and *co-planar*.

The distinction between *larger-smaller inequalities* and *greater-less inequalities* is this: the first refers to absolute magnitude alone, without regard to signs of quality ; the other, in common usage, regards both sign and magnitude.

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FOUR-PLACE LOGARITHMS.

FORM OF A LOGARITHM.

THE LOGARITHM of a number is the exponent of that power to which another number, the *base*, must be raised to give the number first named. The base commonly used is 10; and as most numbers are incommensurable powers of 10, a common logarithm, in general, consists of an integer, the *characteristic*, and an endless decimal, the *mantissa*.

If a number be resolved into two factors, of which one is an integer power of 10 and the other lies between 1 and 10, then the exponent of 10 is the characteristic, and the logarithm of the other factor is the mantissa. The characteristic is positive if the number be larger than 1, and negative if it be smaller; the mantissa is always positive. A negative characteristic is indicated by the sign $-$ above it. The logarithms of numbers that differ only by the position of the decimal point have different characteristics but the same mantissa.

E.g. $7770 = 10^3 \times 7.77$ and $\log 7770 = 3.8904$; $.0777 = 10^{-2} \times 7.77$, and $\log .0777 = \bar{2}.8904$.

TABLES OF LOGARITHMS.

The logarithms of any set of consecutive numbers, arranged in a form convenient for use, constitute a *table of logarithms*. Such a table to the base 10 need give only the mantissas; the characteristics are manifest. This table is arranged upon the common double-entry plan; *i.e.* the mantissa of the logarithm of a three-figure number stands opposite the first two figures and under the third figure. The logarithms are given correct to four places.

TO TAKE OUT THE LOGARITHM OF A NUMBER.

A three-figure number: Take out the tabular mantissa that lies in line with the first two figures of the number and under the third figure; the characteristic is the exponent of that integer power of 10 which lies next below the number.

E.g. $\log 677 = 2.8306$, $\log 6.78 = 0.8312$, $\log .0679 = \bar{2}.8319$, $\log 676\,000 = 5.8299$.

A number of less than three figures: Make the number a three-figure number by annexing zeros, and follow the rule given above.

E.g. $\log 700 = 2.8451$, $\log 7 = 0.8451$, $\log .0071 = \bar{3}.8513$, $\log 71\,000 = 4.8513$.

A four-figure number: Take out the tabular mantissa of the first three figures, and add such part of the difference between this mantissa and the next greater tabular mantissa (the *tabular difference*), as the fourth figure is a part of 10; and so for a five-figure number.

E.g. $\because \log 678 = 2.8312$ and $\log 679 = 2.8319$,

$\therefore \log 678.6 = 2.8312 + .0007 \times 6/10 = 2.8316$, $\log 6.7875 = 0.8312 + .0007 \times 75/100 = 0.8317$.

TO TAKE OUT A NUMBER FROM ITS LOGARITHM.

The mantissa found in the table: Join the figure at the top that lies above the given mantissa to the two figures upon the same line at the extreme left; in this three-figure number so place the decimal point that the number shall be next above that power of 10 whose exponent is the characteristic of the logarithm.

E.g. $\log^{-1} 2.8312 = 678$, $\log^{-1} 0.8451 = 7$, $\log^{-1} \bar{3}.8513 = .0071$, $\log^{-1} 5.8513 = 710\,000$.

The mantissa not found in the table: Take out the three-figure number of the tabular mantissa next less than the given mantissa, and to these three figures join the quotient of the difference of these two mantissas by the tabular difference.

E.g. $\because \log 678 = 2.8312$ and $\log 679 = 2.8319$,

$\therefore \log^{-1} 2.8316 = 678\tfrac{4}{7} = 678.6$, $\log^{-1} \bar{2}.8317 = .0678\tfrac{5}{7} = .06787$.

The use of trigonometric ratios and their logarithms is explained in works on trigonometry.

1	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
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50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0	1	2	3	4	5	6	7	8	9

ANGLE.	SINES. Nat.	SINES. Log.	COSINES. Nat.	COSINES. Log.	TANGENTS. Nat.	TANGENTS. Log.	COTANGENTS. Log.	COTANGENTS. Nat.	ANGLE.
0°00′	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	90°00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	2.5363	343.77	50
20	.0058	7648	1.0000	0000	.0058	7648	2352	171.89	40
30	.0087	9408	1.0000	0000	.0087	9409	0591	114.59	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	1.9342	85.940	20
50	.0145	1627	.9999	0000	.0145	1627	8373	68.750	10
1°00′	0175	8.2419	.9998	9.9999	.0175	8.2419	1.7581	57.290	89°00′
10	.0204	3088	.9998	9999	.0204	3089	6911	49.104	60
20	.0233	3668	.9997	9999	.0233	3669	6331	42.964	40
30	.0262	4179	.9997	9999	.0262	4181	5819	38.188	30
40	.0291	4637	.9996	9998	.0291	4638	5362	84.368	20
50	.0320	5050	.9995	9998	.0320	5053	4947	31.242	10
2°00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	1.4569	28.636	88°00′
10	.0378	5776	.9993	9997	.0378	5779	4221	26.432	50
20	.0407	6097	.9992	9996	.0407	6101	3899	24.542	40
30	.0436	6397	.9990	9996	.0437	6401	3599	22.904	30
40	.0465	6677	.9989	9995	.0466	6682	3318	21.470	20
50	.0494	6940	.9988	9995	.0495	6945	3055	20.206	10
3°00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	1.2806	19.081	87°00′
10	.0552	7423	.9985	9993	.0553	7429	2571	18.075	60
20	.0581	7645	.9983	9993	.0582	7652	2348	17.169	40
30	.0610	7857	.9981	9992	.0612	7865	2135	16.350	30
40	.0640	8059	.9980	9991	.0641	8067	1933	15.605	20
50	.0669	8251	.9978	9990	.0670	8261	1739	14.924	10
4°00′	0698	8.8436	.9976	9.9989	.0699	8.8446	1.1554	14.301	86°00′
10	.0727	8613	.9974	9989	.0729	8624	1376	13.727	60
20	.0756	8783	.9971	9988	.0758	8795	1205	13.197	40
30	.0785	8946	.9969	9987	.0787	8960	1040	12.706	30
40	.0814	9104	.9967	9986	.0816	9118	0882	12.251	20
50	.0843	9256	.9964	9985	.0846	9272	0728	11.826	10
5°00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	1.0580	11.430	85°00′
10	.0901	9545	.9959	9982	.0904	9568	0437	11.059	50
20	.0929	9682	.9957	9981	.0934	9701	0299	10.712	40
30	.0958	9816	.9954	9980	.0963	9836	0164	10.385	30
40	.0987	9945	.9951	9979	.0992	9966	0034	10.078	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	0.9907	9.7882	10
6°00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	0.9784	9.5144	84°00′
10	.1074	0311	.9942	9975	.1080	0336	9664	9.2553	50
20	.1103	0426	.9939	9973	.1110	0453	9547	9.0098	40
30	.1132	0539	.9936	9972	.1139	0567	9433	8.7769	30
40	.1161	0648	.9932	9971	.1169	0678	9322	8.5555	20
50	.1190	0755	.9929	9969	.1198	0786	9214	8.3450	10
7°00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	0.9109	8.1443	83°00′
10	.1248	0961	.9922	9966	.1257	0995	9005	7.9530	50
20	.1276	1060	.9918	9964	.1287	1096	8904	7.7704	40
30	.1305	1157	.9914	9963	.1317	1194	8806	7.5958	30
40	.1334	1252	.9911	9961	.1346	1291	8709	7.4287	20
50	.1363	1345	.9907	9959	.1376	1385	8615	7.2687	10
8°00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	0.8522	7.1154	82°00′
10	.1421	1525	.9899	9956	.1435	1569	8431	6.9682	60
20	.1449	1612	.9894	9954	.1465	1658	8342	6.8269	40
30	.1478	1697	.9890	9952	.1495	1745	8255	0.6912	30
40	.1507	1781	.9886	9950	.1524	1831	8169	6.5606	20
50	.1536	1863	.9881	9948	.1554	1915	8085	6.4348	10
9°00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	0.8003	6.3138	81°00′
ANGLE.	COSINES. Nat.	COSINES. Log.	SINES. Nat.	SINES. Log.	COTANGENTS. Nat.	COTANGENTS. Log.	TANGENTS. Log.	TANGENTS. Nat.	ANGLE.

ANGLE.	SINES.		COSINES.		TANGENTS.		COTANGENTS.		ANGLE.
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
9°00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	0.8003	6.3138	81°00′
10	.1593	2022	.9872	9944	.1614	2078	7922	6.1970	50
20	.1622	2100	.9868	9942	.1644	2158	7842	6.0844	40
30	.1650	2176	.9863	9940	.1673	2236	7764	5.9758	30
40	.1679	2251	.9858	9938	.1703	2313	7687	5.8708	20
50	.1708	2324	.9853	9936	.1733	2389	7611	5.7694	10
10°00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	0.7537	5.6713	80°00′
10	.1765	2468	.9843	9931	.1793	2536	7464	5.5764	50
20	.1794	2538	.9838	9929	.1823	2609	7391	5.4845	40
30	.1822	2606	.9833	9927	.1853	2680	7320	5.3955	30
40	.1851	2674	.9827	9924	.1883	2750	7250	5.3093	20
50	.1880	2740	.9822	9922	.1914	2819	7181	5.2257	10
11°00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	0.7113	5.1446	79°00′
10	.1937	2870	.9811	9917	.1974	2953	7047	5.0658	50
20	.1965	2934	.9805	9914	.2004	3020	6980	4.9894	40
30	.1994	2997	.9799	9912	.2035	3085	6915	4.9152	30
40	.2022	3058	.9793	9909	.2065	3149	6851	4.8430	20
50	.2051	3119	.9787	9907	.2095	3212	6788	4.7729	10
12°00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	0.6725	4.7046	78°00′
10	.2108	3238	.9775	9901	.2156	3336	6664	4.6382	50
20	.2136	3296	.9769	9899	.2186	3397	6603	4.5736	40
30	.2164	3353	.9763	9896	.2217	3458	6542	4.5107	30
40	.2193	3410	.9757	9893	.2247	3517	6483	4.4494	20
50	.2221	3466	.9750	9890	.2278	3576	6424	4.3897	10
13°00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	0.6366	4.3315	77°00′
10	.2278	3575	.9737	9884	.2339	3691	6309	4.2747	50
20	.2306	3629	.9730	9881	.2370	3748	6252	4.2193	40
30	.2334	3682	.9724	9878	.2401	3804	6196	4.1653	30
40	.2363	3734	.9717	9875	.2432	3859	6141	4.1126	20
50	.2391	3786	.9710	9872	.2462	3914	6086	4.0611	10
14°00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	0.6032	4.0108	76°00′
10	.2447	3887	.9696	9866	.2524	4021	5979	3.9617	50
20	.2476	3937	.9689	9863	.2555	4074	5926	3.9136	40
30	.2504	3986	.9681	9859	.2586	4127	5873	3.8667	30
40	.2532	4035	.9674	9856	.2617	4178	5822	3.8208	20
50	.2560	4083	.9667	9853	.2648	4230	5770	3.7760	10
15°00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	0.5719	3.7321	75°00′
10	.2616	4177	.9652	9846	.2711	4331	5669	3.6891	50
20	.2644	4223	.9644	9843	.2742	4381	5619	3.6470	40
30	.2672	4269	.9636	9839	.2773	4430	5570	3.6059	30
40	.2700	4314	.9628	9836	.2805	4479	5521	3.5656	20
50	.2728	4359	.9621	9832	.2836	4527	5473	3.5261	10
16°00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	0.5425	3.4874	74°00′
10	.2784	4447	.9605	9825	.2899	4622	5378	3.4495	50
20	.2812	4491	.9596	9821	.2931	4669	5331	3.4124	40
30	.2840	4533	.9588	9817	.2962	4716	5284	3.3759	30
40	.2868	4576	.9580	9814	.2994	4762	5238	3.3402	20
50	.2896	4618	.9572	9810	.3026	4808	5192	3.3052	10
17°00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	0.5147	3.2709	73°00′
10	.2952	4700	.9555	9802	.3089	4898	5102	3.2371	50
20	.2979	4741	.9546	9798	.3121	4943	5057	3.2041	40
30	.3007	4781	.9537	9794	.3153	4987	5013	3.1716	30
40	.3035	4821	.9528	9790	.3185	5031	4969	3.1397	20
50	.3062	4861	.9520	9786	.3217	5075	4925	3.1084	10
18°00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	0.4882	3.0777	72°00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
ANGLE.	COSINES.		SINES.		COTANGENTS.		TANGENTS.		ANGLE.

ANGLE.	SINES.		COSINES.		TANGENTS.		COTANGENTS.		ANGLE.
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
18°00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	0.4882	3.0777	72°00′
10	.3118	4939	.9502	9778	.3281	5161	4839	3.0475	50
20	.3145	4977	.9492	9774	.3314	5203	4797	3.0178	40
30	.3173	5015	.9483	9770	.3346	5245	4755	2.9887	30
40	.3201	5052	.9474	9765	.3378	5287	4713	2.9600	20
50	.3228	5090	.9465	9761	.3411	5329	4671	2.9319	10
19°00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	0.4630	2.9042	71°00′
10	.3283	5163	.9446	9752	.3476	5411	4589	2.8770	50
20	.3311	5199	.9436	9748	.3508	5451	4549	2.8502	40
30	.3338	5235	.9426	9743	.3541	5491	4509	2.8239	30
40	.3365	5270	.9417	9739	.3574	5531	4469	2.7980	20
50	.3393	5306	.9407	9734	.3607	5571	4429	2.7725	10
20°00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	0.4389	2.7475	70°00′
10	.3448	5375	.9387	9725	.3673	5650	4350	2.7228	50
20	.3475	5409	.9377	9721	.3706	5689	4311	2.6985	40
30	.3502	5443	.9367	9716	.3739	5727	4273	2.6746	30
40	.3529	5477	.9356	9711	.3772	5766	4234	2.6511	20
50	.3557	5510	.9346	9706	.3805	5804	4196	2.6279	10
21°00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	0.4158	2.6051	69°00′
10	.3611	5576	.9325	9697	.3872	5879	4121	2.5826	50
20	.3638	5609	.9315	9692	.3906	5917	4083	2.5605	40
30	.3665	5641	.9304	9687	.3939	5954	4046	2.5386	30
40	.3692	5673	.9293	9682	.3973	5991	4009	2.5172	20
50	.3719	5704	.9283	9677	.4006	6028	3972	2.4960	10
22°00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	0.3936	2.4751	68°00′
10	.3773	5767	.9261	9667	.4074	6100	3900	2.4545	50
20	.3800	5798	.9250	9661	.4108	6136	3864	2.4342	40
30	.3827	5828	.9239	9656	.4142	6172	3828	2.4142	30
40	.3854	5859	.9228	9651	.4176	6208	3792	2.3945	20
50	.3881	5889	.9216	9646	.4210	6243	3757	2.3750	10
23°00′	.3907	9 5919	.9205	9.9640	.4245	9.6279	0.3721	2.3559	67°00′
10	.3934	5948	.9194	9635	.4279	6314	3686	2.3369	50
20	.3961	5978	.9182	9629	.4314	6348	3652	2.3183	40
30	.3987	6007	.9171	9624	.4348	6383	3617	2.2998	30
40	.4014	6036	.9159	9618	.4383	6417	3583	2.2817	20
50	.4041	6065	.9147	9613	.4417	6452	3548	2.2637	10
24°00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	0.3514	2.2460	66°00′
10	.4094	6121	.9124	9602	.4487	6520	3480	2.2286	50
20	.4120	6149	.9112	9596	.4522	6553	3447	2.2113	40
30	.4147	6177	.9100	9590	.4557	6587	3413	2.1943	30
40	.4173	6205	.9088	9584	.4592	6620	3380	2.1775	20
50	.4200	6232	.9075	9579	.4628	6654	3346	2.1609	10
25°00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	0.3313	2.1445	65°00′
10	.4253	6286	.9051	9567	.4699	6720	3280	2.1283	50
20	.4279	6313	.9038	9561	.4734	6752	3248	2.1123	40
30	.4305	6340	.9026	9555	.4770	6785	3215	2.0965	30
40	.4331	6366	.9013	9549	.4806	6817	3183	2.0809	20
50	.4358	6392	.9001	9543	.4841	6850	3150	2.0655	10
26°00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	0.3118	2.0503	64°00′
10	.4410	6444	.8975	9530	.4913	6914	3086	2.0353	50
20	.4436	6470	.8962	9524	.4950	6946	3054	2.0204	40
30	.4462	6495	.8949	9518	.4986	6977	3023	2.0057	30
40	.4488	6521	.8936	9512	.5022	7009	2991	1.9912	20
50	.4514	6546	.8923	9505	.5059	7040	2960	1.9768	10
27°00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	0.2928	1.9626	63°00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
ANGLE.	COSINES.		SINES.		COTANGENTS.		TANGENTS.		ANGLE.

ANGLE.	SINES.		COSINES.		TANGENTS.		COTANGENTS.		ANGLE.
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
27°00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	0.2928	1.9626	63°00′
10	.4566	6595	.8897	9492	.5132	7103	2897	1.9486	50
20	.4592	6620	.8884	9486	.5169	7134	2866	1.9347	40
30	.4617	6644	.8870	9479	.5206	7165	2835	1.9210	30
40	.4643	6668	.8857	9473	.5243	7196	2804	1.9074	20
50	.4669	6692	.8843	9466	.5280	7226	2774	1.8940	10
28°00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	0.2743	1.8807	62°00′
10	.4720	6740	.8816	9453	.5354	7287	2713	1.8676	50
20	.4746	6763	.8802	9446	.5392	7317	2683	1.8546	40
30	.4772	6787	.8788	9439	.5430	7348	2652	1.8418	30
40	.4797	6810	.8774	9432	.5467	7378	2622	1.8291	20
50	.4823	6833	.8760	9425	.5505	7408	2592	1.8165	10
29°00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	0.2562	1.8040	61°00′
10	.4874	6878	.8732	9411	.5581	7467	2533	1.7917	50
20	.4899	6901	.8718	9404	.5619	7497	2503	1.7796	40
30	.4924	6923	.8704	9397	.5658	7526	2474	1.7675	30
40	.4950	6946	.8689	9390	.5696	7556	2444	1.7556	20
50	.4975	6968	.8675	9383	.5735	7585	2415	1.7437	10
30°00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	0.2386	1.7321	60°00′
10	.5025	7012	.8646	9368	.5812	7644	2356	1.7205	50
20	.5050	7033	.8631	9361	.5851	7673	2327	1.7090	40
30	.5075	7055	.8616	9353	.5890	7701	2299	1.6977	30
40	.5100	7076	.8601	9346	.5930	7730	2270	1.6864	20
50	.5125	7097	.8587	9338	.5969	7759	2241	1.6753	10
31°00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	0.2212	1.6643	59°00′
10	.5175	7139	.8557	9323	.6048	7816	2184	1.6534	50
20	.5200	7160	.8542	9315	.6088	7845	2155	1.6426	40
30	.5225	7181	.8526	9308	.6128	7873	2127	1.6319	30
40	.5250	7201	.8511	9300	.6168	7902	2098	1.6212	20
50	.5275	7222	.8496	9292	.6208	7930	2070	1.6107	10
32°00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	0.2042	1.6003	58°00′
10	.5324	7262	.8465	9276	.6289	7986	2014	1.5900	50
20	.5348	7282	.8450	9268	.6330	8014	1986	1.5798	40
30	.5373	7302	.8434	9260	.6371	8042	1958	1.5697	30
40	.5398	7322	.8418	9252	.6412	8070	1930	1.5597	20
50	.5422	7342	.8403	9244	.6453	8097	1903	1.5497	10
33°00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	0.1875	1.5399	57°00′
10	.5471	7380	.8371	9228	.6536	8153	1847	1.5301	50
20	.5495	7400	.8355	9219	.6577	8180	1820	1.5204	40
30	.5519	7419	.8339	9211	.6619	8208	1792	1.5108	30
40	.5544	7438	.8323	9203	.6661	8235	1765	1.5013	20
50	.5568	7457	.8307	9194	.6703	8263	1737	1.4919	10
34°00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	0.1710	1.4826	56°00′
10	.5616	7494	.8274	9177	.6787	8317	1683	1.4733	50
20	.5640	7513	.8258	9169	.6830	8344	1656	1.4641	40
30	.5664	7531	.8241	9160	.6873	8371	1629	1.4550	30
40	.5688	7550	.8225	9151	.6916	8398	1602	1.4460	20
50	.5712	7568	.8208	9142	.6959	8425	1575	1.4370	10
35°00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	0.1548	1.4281	55°00′
10	.5760	7604	.8175	9125	.7046	8479	1521	1.4193	50
20	.5783	7622	.8158	9116	.7089	8506	1494	1.4106	40
30	.5807	7640	.8141	9107	.7133	8533	1467	1.4019	30
40	.5831	7657	.8124	9098	.7177	8559	1441	1.3934	20
50	.5854	7675	.8107	9089	.7221	8586	1414	1.3848	10
36°00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	0.1387	1.3764	54°00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
ANGLE.	COSINES.		SINES.		COTANGENTS.		TANGENTS.		ANGLE.

ANGLE.	SINES.		COSINES.		TANGENTS.		COTANGENTS.		ANGLE.
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
36°00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	0.1387	1.3764	54°00′
10	.5901	7710	.8073	9070	.7310	8639	1361	1.3680	50
20	.5925	7727	.8056	9061	.7355	8666	1334	1.3597	40
30	.5948	7744	.8039	9052	.7400	8692	1308	1.3514	30
40	.6972	7761	.8021	9042	.7445	8718	1282	1.3432	20
50	.5995	7778	.8004	9033	.7490	8745	1255	1.3351	10
37°00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	0.1229	1.3270	53°00′
10	.6041	7811	.7969	9014	.7581	8797	1203	1.3190	50
20	.6065	7828	.7951	9004	.7627	8824	1176	1.3111	40
30	.6088	7844	.7934	8995	.7673	8850	1150	1.3032	30
40	.6111	7861	.7916	8985	.7720	8876	1124	1.2954	20
50	.6134	7877	.7898	8975	.7766	8902	1098	1.2876	10
38°00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	0.1072	1.2799	52°00′
10	.6180	7910	.7862	8955	.7860	8954	1046	1.2723	50
20	.6202	7926	.7844	8945	.7907	8980	1020	1.2647	40
30	.6225	7941	.7826	8935	.7954	9006	0994	1.2572	30
40	.6248	7957	.7808	8925	.8002	9032	0968	1.2497	20
50	.6271	7973	.7790	8915	.8050	9068	0942	1.2423	10
39°00′	.8293	9.7989	.7771	9.8905	.8098	9.9084	0.0916	1.2349	51°00′
10	.6316	8004	.7753	8895	.8146	9110	0890	1.2276	50
20	.8338	8020	.7735	8884	.8195	9135	0865	1.2203	40
30	.6361	8035	.7716	8874	.8243	9161	0839	1.2131	30
40	.8383	8050	.7698	8864	.8292	9187	0813	1.2059	20
50	.6406	8066	.7679	8853	.8342	9212	0788	1.1988	10
40°00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	0.0762	1.1918	50°00′
10	.6450	8096	.7642	8832	.8441	9264	0736	1.1847	50
20	.6472	8111	.7623	8821	.8491	9289	0711	1.1778	40
30	.6494	8125	.7604	8810	.8541	9315	0685	1.1708	30
40	.6517	8140	.7585	8800	.8591	9341	0659	1.1640	20
50	.6539	8155	.7566	8789	.8642	9366	0634	1.1571	10
41°00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	0.0608	1.1504	49°00′
10	.6583	8184	.7528	8767	.8744	9417	0583	1.1436	50
20	.6604	8198	.7509	8756	.8796	9443	0557	1.1369	40
30	.6626	8213	.7490	8745	.8847	9468	0532	1.1303	30
40	.6648	8227	.7470	8733	.8899	9494	0506	1.1237	20
50	.6670	8241	.7451	8722	.8952	9519	0481	1.1171	10
42°00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	0.0456	1.1106	48°00′
10	.8713	8269	.7412	8699	.9057	9570	0430	1.1041	50
20	.8734	8283	.7392	8688	.9110	9595	0405	1.0977	40
30	.6756	8297	.7373	8676	.9163	9621	0379	1.0913	30
40	.6777	8311	.7353	8666	.9217	9646	0354	1.0850	20
50	.6799	8324	.7333	8653	.9271	9671	0329	1.0786	10
43°00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	0.0303	1.0724	47°00′
10	.6841	8351	.7294	8699	.9380	9722	0278	1.0661	50
20	.6862	8365	.7274	8618	.9435	9747	0253	1.0599	40
30	.6884	8378	.7254	8606	.9490	9772	0228	1.0538	30
40	.6905	8391	.7234	8594	.9545	9798	0202	1.0477	20
60	.6926	8405	.7214	8582	.9601	9823	0177	1.0416	10
44°00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	0.0152	1.0355	46°00′
10	.6967	8431	.7173	8557	.9713	9874	0126	1.0295	50
20	.6088	8444	.7153	8545	.9770	9899	0101	1.0235	40
30	.7009	8457	.7133	8532	.9827	9924	0076	1.0176	30
40	.7030	8469	.7112	8520	.9884	9949	0051	1.0117	20
50	.7050	8482	.7092	8507	.9942	9075	0025	1.0068	10
45°00′	.7071	9.8495	.7071	0.8495	1.0000	0.0000	0.0000	1.0000	45°00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Log.	Nat.	
ANGLE.	COSINES.		SINES.		COTANGENTS.		TANGENTS.		ANGLE.

TRIGONOMETRY.

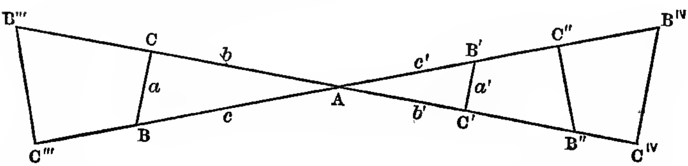
TRIGONOMETRY is that branch of mathematics which treats of the numerical relations of angles and triangles. It is essentially algebraic in character, but is founded on geometry.

I. THE RIGHT TRIANGLE.

§1. TRIGONOMETRIC RATIOS.

THEOR. 1. *If from a point in one side of an acute angle a perpendicular fall on the other side, then, in the right triangle so formed, the ratio of the side opposite the angle to the hypotenuse is the same, whatever point be taken; and so for that of the adjacent side to the hypotenuse, for that of the opposite side to the adjacent side, and for the reciprocals of these three ratios.*

For let A be any acute angle; B, B$'\cdots$ points on either bounding line; $a, a'\cdots$ perpendiculars from B, B$'\cdots$ to the other line at C, C$'\cdots$; $b, b'\cdots$ the lines AC, AC$'\cdots$; and $c, c'\cdots$ the lines AB, AB$'\cdots$;



then $\because$ the right triangles ABC, AB$'$C$'\cdots$ are similar,

$\therefore$ the ratios $a/c,\ a'/c'\cdots$ are all equal;

and so for the other ratios $b/c,\ b'/c'\cdots,\quad a/b,\ a'/b'\cdots,$

$b/a,\ b'/a'\cdots,\quad c/b,\ c'/b'\cdots,\quad c/a,\ c'/a'\cdots.$

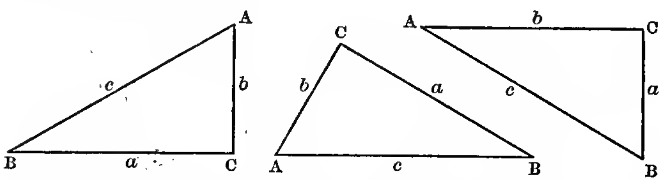
But if an angle be taken greater or less than A, the triangles so formed are not similar to these, and the ratios are different from those for the angle A.

For this reason an acute angle has its six ratios distinct from the ratios of every other angle, and if one of the ratios be given the angle can be constructed. These ratios are the six principal *trigonometric functions* of an angle, and they are named as follows:

opposite side to hypotenuse, the *sine* of the angle,
adjacent side to hypotenuse, the *cosine*,
opposite side to adjacent side, the *tangent*,
adjacent side to opposite side, the *cotangent*,
hypotenuse to adjacent side, the *secant*,
hypotenuse to opposite side, the *cosecant*.

When written before the name of the angle, the words sine, cosine, tangent, cotangent, secant, cosecant may be abbreviated to sin, cos, tan, cot, sec, csc. Standing alone, the abbreviations have no meaning.

If ABC be any right triangle with C the right angle, a the side opposite the acute angle A, b the side opposite the acute angle B, and c the hypotenuse, then the six ratios of each of the acute angles may be expressed in terms of the three sides of the triangle, as below.



$\sin \mathrm{A} = a/c$, $\csc \mathrm{A} = c/a$, and $\sin \mathrm{B} = b/c$, $\csc \mathrm{B} = c/b$,
$\cos \mathrm{A} = b/c$, $\sec \mathrm{A} = c/b$, $\cos \mathrm{B} = a/c$, $\sec \mathrm{B} = c/a$,
$\tan \mathrm{A} = a/b$, $\cot \mathrm{A} = b/a$, $\tan \mathrm{B} = b/a$, $\cot \mathrm{B} = a/b$.

NOTE. In the discussion of the right triangle that follows, the triangle is always lettered as in the figures above; *i.e.*,

with C for the right angle, c for the hypotenuse, A, B for the acute angles, a, b for the sides opposite A, B.

The expression $\sin^{-1} a/c$ means the angle whose sine is a/c; $\cos^{-1} b/c$, the angle whose cosine is b/c; $\tan^{-1} a/b$, the angle whose tangent is a/b, and so for the other ratios. They are read: the *anti-sine* of a/c, the *anti-cosine* of b/c, the *anti-tangent* of a/b, and so on.

E.g. if $\sin A = \frac{1}{2}$, if $\cos B = \frac{3}{5}$, if $\tan F = 6$, if $\cot X = \sqrt{3}$,
then $A = \sin^{-1} \frac{1}{2}$, $B = \cos^{-1} \frac{3}{5}$, $F = \tan^{-1} 6$, $X = \cot^{-1} \sqrt{3}$.

The index $^{-1}$ is to be carefully distinguished from the negative exponent; it is analogous to that in the expression $\log^{-1} 2$, which is read the *anti-logarithm* of 2 and means the number whose logarithm is 2.

The positive powers of the trigonometric ratios are commonly written in the form $\sin^2 A$, $\cos^3 B$, instead of $(\sin A)^2$, $(\cos B)^3$; but their reciprocals are written in the form of fractions, or with the exponent without the bracket.

E.g. $1/\sin A$, or $(\sin A)^{-1}$, not $\sin^{-1} A$; $1/\cos^2 B$, or $(\cos B)^{-2}$.

QUESTIONS.

1. If the sides a, b, c, of a right triangle ABC be 3 feet, 4 feet, 5 feet, what are the six ratios of A and of B? if the sides be 3 yards, 4 yards, 5 yards? if 3 miles, 4 miles, 5 miles?

2. In a right triangle ABC the sides a, c are 12 yards and 13 yards: find b and the six ratios of A and of B.

So, if a, b be 12 feet and 5 feet, find c and the six ratios.

3. Construct the right triangle ABC with the hypotenuse c 5 feet, and a side a 3 feet. What is the sine of the angle A?

From this construct a right triangle ABC if $\sin A = \frac{3}{5}$.

So, if $\cos A = \frac{4}{5}$, if $\tan A = \frac{3}{4}$, if $\cot A = \frac{4}{3}$, if $\sec A = \frac{5}{4}$.

4. Construct $\sin^{-1} \frac{1}{2}$, $\cos^{-1} \frac{3}{4}$, $\tan^{-1} \frac{4}{3}$, $\cot^{-1} \frac{3}{4}$, $\sec^{-1} \frac{4}{3}$.

5. Find the six ratios of one of the acute angles of a right isosceles triangle.

6. Draw a perpendicular from the vertex to the base of an equilateral triangle, and find the six ratios of the acute angles of the right triangles so formed.

7. In a right triangle ABC, let the hypotenuse c be 12 feet and the angle A be 30°: find the sides a, b, given $\sin 30° = .5$, $\cos 30° = .866$, nearly.

8. In a right triangle ABC, let the side a be 12 yards and the angle A be 35° : find the sides b, c, given $\sin 35° = .574$, $\tan 35° = .7$.

9. In a right triangle ABC, let the side b be 12 miles and the angle A be 40°: find the sides c, a, given $\cos 40° = .766$, $\cot 40° = 1.192$.

10. In a right triangle ABC, let the hypotenuse c be 12 feet and the side a be 8.484 feet: find the side b and the angle A, given $\sin 45° = .707$.

11. In a right triangle ABC, let the side a be 12 yards and the side b be 9.948 yards : find the side c and the angle A, given $\sin 50° = .766$, $\tan 50° = 1.192$.

12. In a right triangle ABC, let the side b be 12 miles and the hypotenuse c be 20.9 miles : find the side a and the angles, given $\cos 55° = .574$, $\tan 55° = 1.428$.

13. In a right triangle ABC, let the side a be 12 metres and the hypotenuse c be $33\frac{1}{3}$ metres: find the side b and the angles, given $\sin 21° 6' = .36$, $\cos 21° 6' = .933$.

Verify the work by showing that $a^2 + b^2 = c^2$.

14. Draw two right triangles ABC, A′B′C′, having A larger than A′, and show which of the ratios of A are larger, and which are smaller, than the like-named ratios of A′.

15. Draw a right triangle having an acute angle less than half a right angle, and show which of the ratios of that angle are larger than unity, and which are smaller.

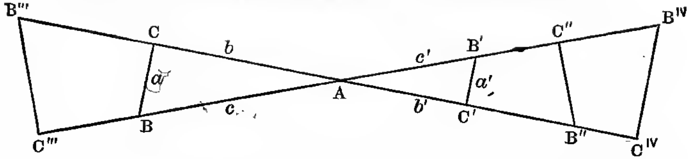
16. Draw a right triangle having one acute angle very small, and show which of the ratios of this angle are very small, which are very large, and which are near unity.

As the angle is made smaller and smaller, approaching zero, to what do these ratios approach ?

So, what are the ratios of the other acute angle, which is very near a right angle ?

THEOR. 2. *If* A *be any acute angle, then:*

$$\sin A \cdot \csc A = 1, \quad \cos A \cdot \sec A = 1, \quad \tan A \cdot \cot A = 1.$$



For, from any point B of either side of the angle, let fall a perpendicular BC upon the other side, as in theor. 1;
then ∵ in the right triangle ABC so formed,

$\sin A = a/c, \quad \csc A = c/a,$ [df.

$\therefore \sin A \cdot \csc A = a/c \cdot c/a = 1.$ Q.E.D.

So, $\because \cos A = b/c, \quad \sec A = c/b,$ [df.

$\therefore \cos A \cdot \sec A = b/c \cdot c/b = 1.$ Q.E.D.

So, $\because \tan A = a/b, \quad \cot A = b/a,$ [df.

$\therefore \tan A \cdot \cot A = a/b \cdot b/a = 1.$ Q.E.D.

THEOR. 3. *If* A *be any acute angle, then:*

$$\sin A/\cos A = \tan A, \quad \cos A/\sin A = \cot A.$$

For ∵ in the right triangle ABC, formed as in theor. 1,

$\sin A = a/c, \quad \cos A = b/c, \quad \tan A = a/b, \quad \cot A = b/a,$ [df

$\therefore \sin A/\cos A = a/c : b/c = a/b = \tan A,$

and $\cos A/\sin A = b/c : a/c = b/a = \cot A.$ Q.E.D.

THEOR. 4. *If* A *be any acute angle, then:*

$$\sin^2 A + \cos^2 A = 1, \quad 1 + \tan^2 A = \sec^2 A, \quad 1 + \cot^2 A = \csc^2 A.$$

For ∵ in the right triangle ABC, $a^2 + b^2 = c^2,$

$\therefore a^2/c^2 + b^2/c^2 = 1;$ [div. by c^2.

and $\because \sin A = a/c, \quad \cos A = b/c,$ [df.

$\therefore \sin^2 A + \cos^2 A = 1.$ Q.E.D.

So, $a^2/b^2 + 1 = c^2/b^2;$ [div. by b^2.

and $\because \tan A = a/b, \quad \sec A = c/b,$ [df.

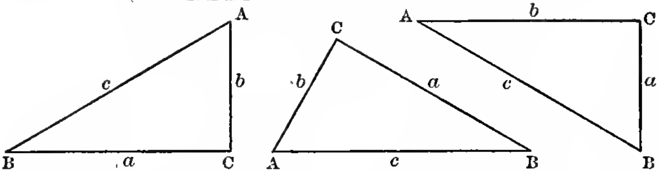
$\therefore 1 + \tan^2 A = \sec^2 A.$ Q.E.D.

So, $1 + b^2/a^2 = c^2/a^2;$ [div. by a^2.

and $\because \cot A = b/a, \quad \csc A = c/a,$

$\therefore 1 + \cot^2 A = \csc^2 A.$ Q.E.D.

THEOR. 5. *If* A *be any acute angle and* B *the complement of* A, *then:* $\sin A = \cos B$, $\tan A = \cot B$, $\sec A = \csc B$.



For ∵ in the right triangle ABC, formed as in theor. 1, A, B are complementary acute angles,

and ∵ $\sin A = a/c$, and $\cos B = a/c$, [df.

∴ $\sin A = \cos B$. Q.E.D.

So, ∵ $\tan A = a/b$, and $\cot B = a/b$, [df.

∴ $\tan A = \cot B$. Q.E.D.

So, ∵ $\sec A = c/b$, and $\csc B = c/b$, [df.

∴ $\sec A = \csc B$. Q.E.D.

NOTE. If the sine, tangent, and secant of an angle be called its *direct ratios*, and the cosine, cotangent, and cosecant the *co-ratios*, theor. 5 may be stated as follows: *the direct ratios of an angle are the co-ratios of its complement.*

The words cosine, cotangent, and cosecant are but abbreviated forms for complement-sine, complement-tangent, and complement-secant; *i.e.* for sine of complement, tangent of complement, and secant of complement.

QUESTIONS.

1. Translate the equation $\sin^2 A + \cos^2 A = 1$ into words, and express its meaning as a theorem.

Solve this equation in turn for sin A and cos A, and translate the resulting equations into theorems.

2. Translate the equation $\sec^2 A = 1 + \tan^2 A$ into words, and express its meaning as a theorem.

Solve this equation in turn for sec A and tan A, and translate the resulting equations into theorems.

So, the equation $\csc^2 A = 1 + \cot^2 A$.

3. Show that

$$\sin A = \tan A \cdot \cos A = \tan A/\sec A = \cos A/\cot A,$$
$$\csc A = \sec A/\tan A = \cot A/\cos A = \cot A \cdot \sec A,$$
$$\cos A = \cot A \cdot \sin A = \cot A/\csc A = \sin A/\tan A,$$
$$\sec A = \csc A/\cot A = \tan A/\sin A = \tan A \cdot \csc A,$$
$$\tan A = \sin A \cdot \sec A = \sec A/\csc A, = \sin A/\cos A$$
$$\cot A = \cos A \cdot \csc A = \csc A/\sec A. = \cos A/\sin A$$

Translate these equations into theorems.

4. Show that

$$\sin A = \tan A/\sqrt{(\tan^2 A + 1)} = \sqrt{(\sec^2 A - 1)}/\sec A,$$
$$\csc A = \sqrt{(\tan^2 A + 1)}/\tan A = \sec A/\sqrt{(\sec^2 A - 1)},$$
$$\cos A = \cot A/\sqrt{(\cot^2 A + 1)} = \sqrt{(\csc^2 A - 1)}/\csc A,$$
$$\sec A = \sqrt{(\cot^2 A + 1)}/\cot A = \csc A/\sqrt{(\csc^2 A - 1)},$$
$$\tan A = \sin A/\sqrt{(1 - \sin^2 A)} = \sqrt{(1 - \cos^2 A)}/\cos A,$$
$$\cot A = \sqrt{(1 - \sin^2 A)}/\sin A = \cos A/\sqrt{(1 - \cos^2 A)}.$$

Translate these equations into theorems.

5. If the hypotenuse c of a right triangle ABC have unit length, show that the two legs a, b, have the lengths $\sin A$, $\sqrt{(1 - \sin^2 A)}$, and thence find the values of $\tan A$, $\cot A$, $\sec A$, in terms of $\sin A$.

So, show that the two legs a, b, have the lengths $\sqrt{(1 - \cos^2 A)}$, $\cos A$, and thence find the values of $\tan A$, $\cot A$, $\csc A$, in terms of $\cos A$.

6. If the leg b of a right triangle ABC have unit length, show that the leg a and hypotenuse c have the lengths $\tan A$, $\sqrt{(\tan^2 A + 1)}$, and thence find the values of $\sin A$, $\cos A$, $\sec A$, $\csc A$, in terms of $\tan A$.

7. With the values of the ratios of the angles 30°, 45°, 60°, as found in examples 5, 6, page 3, find the values of A from the equations: $\tan A + \cot A = 2,$ 45 $\sin A + \cos A = \sqrt{2},$ 115
$\sin A \cdot \sec A = \sqrt{3},$ 60 $\cot A = 2 \cos A.$ 30

8. Find the other ratios of A

if $\sin A = \frac{3}{5}$, if $\cos A = \frac{4}{5}$, if $\tan A = \frac{3}{4}$, if $\cot A = \frac{5}{12}$.

§ 2. TRIGONOMETRIC TABLES.

The magnitude of an angle is commonly expressed in degrees, minutes, and seconds, *e.g.* $68° 25' 30''$. A degree is the ninetieth part of a right angle; a minute, the sixtieth part of a degree; a second, the sixtieth part of a minute.

In the computation of triangles and generally in operations that involve angles, the angles themselves play no direct part, but the six trigonometric ratios are always used. By methods to be explained later, these ratios have been computed for different angles and arranged in tables for convenient use.

In the small tables (pp. IX–XVI) both the ratios themselves, the *natural functions,* and their logarithms, the *logarithmic functions,* are given correct to four figures for angles differing by ten minutes, from 0° to 90°. If a logarithm be negative, 10 is added and the *modified logarithm* is given.

The two angles printed on one line are complementary angles, and the direct functions of the one are the co-functions of the other. Angles less than 45° are found at the left side of the page, and the names of their functions at the top; angles greater than 45° are at the right side, and the names of their functions at the bottom.

The functions of an angle given in the tables may be read directly from the tables; but those of an angle not so given are found from those of the next less and next greater tabular angles, on the principle that small differences of angles and the corresponding small differences of functions, are very nearly proportional.

E.g. $\because \sin 25° = .4226, \quad \sin 25° 10' = .4252$, nearly, [table.
and $\sin 25° 5'$ lies midway between $\sin 25°$ and $\sin 25° 10'$,
$\therefore \sin 25° 5' = .4239$, nearly.

So, $\because$ log-tan $25° 20' = 9.6752$, log-tan $25° 30' = 9.6785$,
$\therefore$ log-tan $25° 22' = 9.6752 + \frac{2}{10}(9.6785 - 9.6752) = 9.6759$.

If the functions be given in the table, then the angles may be read directly; but if not so given they may be found from the next less and next greater tabular functions.

E.g. $\because \cos^{-1}.4252 = 64° \, 50', \quad \cos^{-1}.4226 = 65°,$ [table.

$\therefore \cos^{-1}.4239 = 64° \, 50' + \frac{13}{26} \cdot 10' = 64° \, 55'.$

So, $\because$ log-cot^{-1} 9.6785 = 64° 30′, log-cot^{-1} 9.6752 = 64° 40′,

$\therefore$ log-cot^{-1} 9.6759 = 64° 40′ − $\frac{7}{33} \cdot 10'$ = 64° 38′.

In the larger tables the decimals are carried to five, or six, or seven places, the ratios are given for angles that differ by one minute, or by ten seconds, or by one second, and there are many labor-saving devices.

Of these devices, the most common is that of printing the differences of consecutive logarithmic sines in a column at the right of the column of sines, that of cosines at the right of the column of cosines, and that of tangents and cotangents (the same differences for both) between the columns of tangents and cotangents. These differences are called the tabular differences.

QUESTIONS.

From the table of natural functions, find :

1. sin 20°, 21°, 20° 10′, 20° 20′, 79° 18′, 57° 15′.
2. cos 20°, 21°, 20° 10′, 20° 20′, 79° 18′, 57° 15′.
3. tan 35°, 36°, 35° 15′, 35° 25′, 79° 58′, 25° 36′.
4. cot 35°, 36°, 35° 15′, 35° 25′, 79° 58′, 25° 36′.

From the table of logarithmic functions, find :

5. log-sin 20°, 21°, 20° 10′, 20° 20′, 79° 18′, 57° 15′.
6. log-cos 20°, 21°, 20° 10′, 20° 20′, 79° 18′, 57° 15′.
7. log-tan 35°, 36°, 35° 15′, 35° 25′, 79° 58′, 25° 36′.
8. log-cot 35°, 36°, 35° 15′, 35° 25′, 79° 58′, 25° 36′.

From the table of natural functions, find :

9. sin^{-1} .2588, .2591, .2590, .9279, .9281, .9280.
10. cos^{-1} .9279, .9281, .9280, .2591, .2588, .2590.
11. tan^{-1} .5022, .5059, .5035, .9217, .9271, .9250.
12. cot^{-1} .9217, .9271, .9250, .5022, .5059, .5035.

From the table of logarithmic functions, find :

13. log-sin^{-1} 8.5809, 8.5842, 8.5821, 9.9997, 9.9847.
14. log-cos^{-1} 8.5809, 8.5842, 8.5821, 9.9997, 9.9847.
15. log-tan^{-1} 8.5812, 8.5845, 8.5831, 1.4188, 1.3071.
16. log-cot^{-1} 8.5812, 8.5845, 8.5831, 1.4188, 1.3071.

§ 3. THE SOLUTION OF RIGHT TRIANGLES.

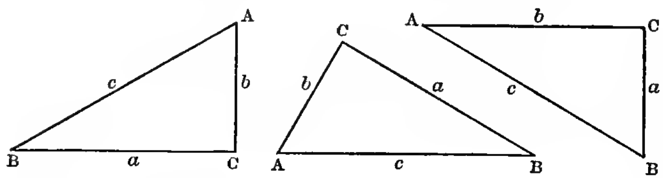
Three parts, one being a side, are sufficient to determine a plane triangle; and the *solution* of a triangle consists in finding the three unknown parts from the three that are given.

In a right triangle if a side and one other part be known, the triangle may be solved by forming equations that involve the two known parts and one unknown part, and solving these equations for the unknown parts.

In general the work may be *checked* by forming independent equations that involve the three computed parts, and which cannot be true unless the work be correct.

Let ABC be a right triangle then, whatever parts be given, all the equations needed are found among these:

$$A+B=90^\circ, \quad a^2+b^2=c^2,$$
$$\sin A=a/c, \quad \cos A=b/c, \quad \tan A=a/b,$$
$$\sin B=b/c, \quad \cos B=a/c, \quad \tan B=b/a.$$



There are four cases:

1. Given c, A, the hypotenuse and an acute angle:
then $B=90^\circ-A$, $a=c\cdot\sin A$, $b=c\cdot\cos A$.
Checks: $\tan B=b/a$, $b^2=(c+a)(c-a)$, $a^2=(c+b)(c-b)$.

2. Given b, A, a side and an acute angle:
then $B=90^\circ-A$, $c=b/\cos A$, $a=b\cdot\tan A$.
Checks: $\cos B=a/c$, $b^2=(c+a)(c-a)$, $a^2=(c+b)(c-b)$.

3. Given c, b, the hypotenuse and a side:
then $\cos A=b/c$, $B=90^\circ-A$, $a=b\cdot\tan A$.
Checks: $\cos B=a/c$, $b^2=(c+a)(c-a)$, $a^2=(c+b)(c-b)$.

4. Given a, b, the two sides about the right angle:
then $\tan A=a/b$, $B=90^\circ-A$, $c=b/\cos A$.
Checks: $\cos B=a/c$, $b^2=(c+a)(c-a)$, $a^2=(c+b)(c-b)$.

E.g. let $c=125$, $A=40°$; then $B=90°-40°=50°$;
and, with natural functions, the work may take this form:

$$\begin{array}{rl}
\sin 40° = & .6428 \\
& \times 125 \\
a = & \overline{80.35.}
\end{array}
\qquad\qquad
\begin{array}{rl}
\cos 40° = & .7660 \\
& \times 125 \\
b = & \overline{95.75.}
\end{array}$$

check: $\tan b = b/a$, $\qquad (c+b)(c-b)=a^2$

$$80.35\,)\,95.75\,(\,1.1917$$

$$\begin{array}{rl}
c = & 125 \\
b = & 95.75 \\
c+b = & \overline{220.75} \\
c-b = & 29.25\times \\
c^2-b^2 = & \overline{6456.9375}
\end{array}
\qquad
\begin{array}{rl}
a = & 80.35 \\
& \times 80.35 \\
a^2 = & \overline{6456.1225}
\end{array}$$

$$\tan 50° = 1.1918$$

So, with logarithmic functions, the work may take this form:

$$\begin{array}{rl}
\text{log-sin } 40° = & 9.8081 \\
\log 125 = & 2.0969+ \\
\log a = & \overline{1.9050}, \quad a=80.35
\end{array}
\qquad
\begin{array}{rl}
\text{log-cos } 40° = & 9.8843 \\
\log 125 = & 2.0969+ \\
\log b = & \overline{1.9812}, \quad b=95.76
\end{array}$$

check:

$$\begin{array}{rll}
c = & 125 & \\
b = & 95.76 & \\
c+b = & \overline{220.76} & \log, 2.3439 \\
c-b = & 29.24 & 1.4660 \\
& \log(c^2-b^2) = & \overline{3.8099.}
\end{array}
\qquad
\begin{array}{rl}
\log a = & 1.9050 \\
& \times 2 \\
\log a^2 = & \overline{3.8100.}
\end{array}$$

NOTE. The two solutions do not quite agree, and the checks are not perfect; the defects arise from the use of the small tables. More exact results come from larger tables, that give the ratios correct to five, six, or seven figures.

QUESTIONS.

Solve these right triangles, using natural functions, given:

1. c, 40 yds.; A, 30°.
2. c, 12.5 ft.; B, 68° 10′.
3. b, 187 metres; A, 55° 20′.
4. a, 7.57 in.; B, 9° 30′.
5. b, 18.5 ft.; c, 125 ft.
6. c, 37 mi.; a, 25.2 mi.
7. a, 59.3 yds.; b, 45.7 yds.
8. a, 4 ft. 6 in.; b, 12 ft. 9. in.

Solve these right triangles, using logarithmic functions, given:

a=110

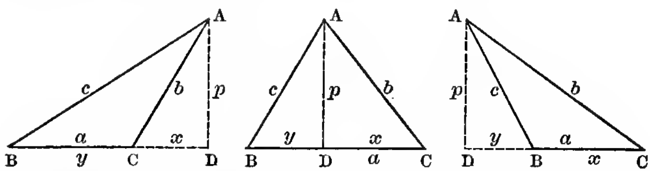
9. c, 127 ft.; A, 60°.
10. c, 18.7 yds.; B, 76° 15′.
11. b, 45.9 yds.; A, 59° 15′.
12. a, 18.3 chs.; B, 55° 12′.
13. b, 597 m.; c, 676 m.
14. a, 1278 yds.; c, 1355 yds.
15. a, 27.85 in.; b, 5519 in.
16. a, 8539 ft.; b, 2815 ft.

§ 4. ISOSCELES AND OBLIQUE TRIANGLES.

In an isosceles triangle, the perpendicular from the vertex to the base divides the triangle into two equal right triangles; and if two parts of one of these triangles be given, this triangle may be solved, and so the whole triangle is solved.

If three parts of an oblique triangle be given, always including a side, a perpendicular may fall from a vertex to the opposite side and so divide the given triangle into two right triangles, and by their solution the triangle is solved.

Let ABC be any oblique triangle, a, b, c the sides opposite the angles A, B, C; p the perpendicular AD from A to a; x, y the segments CD, BD, of a.



The statements below apply directly to the second of the three figures; but with slight modifications suggested by the figures themselves, they apply to the other figures as well.

There are four cases:

1. Given a, b, c, the three sides:

then $\because p^2+x^2=b^2, \quad p^2+y^2=c^2,$

$\therefore x^2-y^2=b^2-c^2;$

and $\because x+y=a,$

$\therefore x-y=(b^2-c^2)/a,$

$\therefore x=\frac{1}{2}[a+(b^2-c^2)/a]=(a^2+b^2-c^2)/2a,$

and $\quad y=\frac{1}{2}[a-(b^2-c^2)/a]=(a^2-b^2+c^2)/2a;$

and two parts of each right triangle are known.

2. Given b, B, C, a side and two angles:

then, in the right triangle ACD, b and C are known, and p and x may be computed;

and, in the right triangle ABD, p and B are known, and c and y may be computed.

$a=x+y, \quad A=180°-(B+C).$

3. Given c, a, B, two sides and the included angle:
then, in the right triangle ABD, c and B are known, and p and y may be computed;
and, in the right triangle ACD, p is known, $x=a-y$, and b and C may be computed.
$A=180°-(B+C)$.

4. Given b, c, B, two sides and an opposite angle:
then, in the right triangle ABD, c and B are known, and p and y may be computed;
and, in the right triangle ACD, b and p are known, and x and C may be computed.
$a=y\pm x$, $A=180°-(B+C)$.

QUESTIONS.

Solve these isosceles triangles, given:

1. The sides 10 yards, and the base 16 yards.
2. The vertical angle 90°, and the base 10 yards.
3. The base 10 yards, and the base angles 70°.
4. The vertical angle 70°, and a side 12 yards.
5. The base 18 yards, and a side 12 yards.

6. If two sides and an angle opposite one of them be given, b, c, B, the side c is given in length and position both, a in position but not in length, b in length but not in position, and b finds its position only as it swings about A as a hinge till its lower end rests on the line of the base: if then the angle B be acute, and if the swinging side b be shorter than the perpendicular p, is a triangle possible? is there a triangle if b be just as long as p? of what kind is it? is there one triangle or two if b be longer than p, but shorter than c? if b be just as long as c? if b be longer than c? Draw figures to illustrate.

So, if B be right or obtuse?

Solve these oblique triangles, given:

7. a, 13; b, 15; c, 17.
8. a, 357; b, 537; c, 735.
9. c, 5; a, 7; B, 65°.
10. a, 537; b, 753; C, 119° 15′.
11. b, 30; B, 55°; C, 48° 25′.
12. a, 7.5; A, 84°; B, 42° 37′.
13. b, 5, 10, 15, 20, 25 in turn; c, 20; B, 30°.

§ 5. HEIGHTS AND DISTANCES.

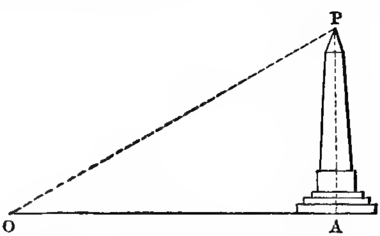
The *plane of the horizon* at any point on the earth's surface is the plane that is tangent to the surface, *i.e.* to the surface of still water, at that point; it would therefore be perpendicular to the radius of the earth, if the earth were a perfect sphere. The direction perpendicular to the horizon-plane is determined by a plumb line; it is a *vertical line,* and any plane containing this line is a *vertical plane.* Any plane parallel to the horizon-plane is a *horizontal plane,* and such a plane may be determined by a spirit level.

An angle lying in a horizontal plane is a *horizontal angle,* and an angle lying in a vertical plane is a *vertical angle.* The vertical angle made with the horizontal plane by the line of sight from the observer to any object is its *angle of elevation* if the object be above the observer, and its *angle of depression* if it be below him.

Ordinary field instruments measure horizontal and vertical angles only. By *distance* is meant the horizontal distance, unless otherwise named; and by *height* is meant the vertical distance of a point above or below the plane of observation. A surveyor's chain is four rods long and it is divided into a hundred *links.* Ten square chains make an acre.

To find the height above its base of a vertical column, AP, *whose base is accessible.*

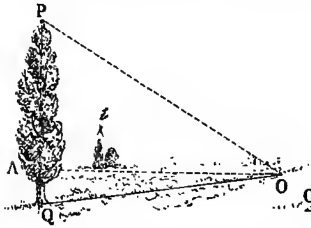
1. If the column AP stand on a horizontal plane:

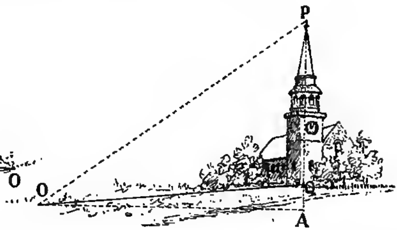


From the base A measure any convenient distance AO, and the angle AOP;

and solve the right triangle AOP, for AP.

2. If the column PQ stand on an inclined plane:





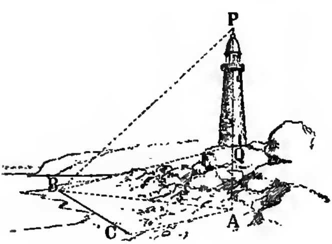
Let P be the top of the column, Q the point at the base of the column below P, and A a point below P in the horizontal plane through the point of observation, O;

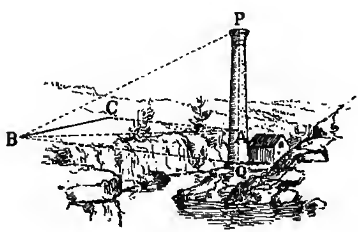
measure any convenient distance QO along the plane, and the angles of elevation or depression AOP, AOQ;

solve the right triangles AOP, AOQ: then $PQ = AP \pm AQ$.

To find the distance from the observer, and the height above its base, of an inaccessible but visible vertical column.

Let P be the top of the column, Q the base, B the position of the observer, A the point vertically below P in the horizontal plane through B;





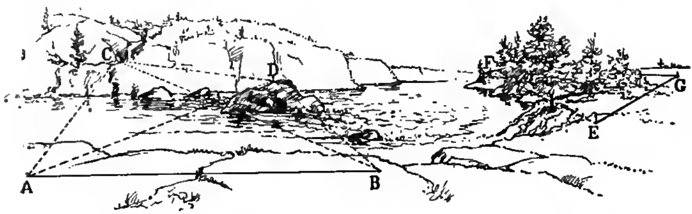
take any other convenient point of observation C, and measure the horizontal line BC, the horizontal angles ABC, ACB, and the vertical angles ABP, ABQ;

solve the horizontal oblique triangle ABC for AB, and the vertical right triangles ABP, ABQ for AP, AQ: then $PQ = AP \pm AQ$.

If the observer be in the same horizontal plane as the base, the line BQ coincides with BA, and BAP is the only vertical triangle to be computed.

To find the distance apart of two objects that are separated by an impassable barrier.

1. If both objects be accessible :



Let E, F be the two objects, and G the point of observation ; measure the horizontal lines GE, GF and the horizontal angle EGF, and compute EF.

2. If both objects be inaccessible :

Let C, D be the two objects ; measure any convenient line AB and the horizontal angles ABC, ABD, BAC, BAD ;

in triangle ABD compute BD ; in ABC compute BC ; in BCD compute CD.

This is the method of *triangulation ;* AB is the *base line.*

QUESTIONS.

1. At 120 feet distance, and on a level with the foot of a steeple, the angle of elevation of the top is 62° 27′ : find the height. [230.03 feet.

2. From the top of a rock 326 feet above the sea, the angle of depression of a ship's hull is 25° 42′ : find the distance of the ship. [677.38 feet.

3. A ladder $29\frac{1}{2}$ feet long standing in the street just reaches a window 25 feet high on one side of the street, and 23 feet high on the other side : how wide is the street ? [34.13 feet.

4. From the top of a hill I observe two successive milestones in the plain below, and in a straight line before me, and find their angles of depression to be 5° 30′, 14° 20′ : what is the height of the hill? [815.85 feet.

5. Two observers on the same side of a balloon, and in the same vertical plane with it, are a mile apart, and find the angles of elevation to be 17° and 68° 25′ respectively: what is its height? [1836 feet.

6. From the top of a mountain $1\frac{1}{2}$ miles high, the *dip* of the sea-horizon (angle of depression of sky-and-water line) is 1° 34′ 40″ : find the earth's diameter, and the distance of the sea-horizon.

7. What is the distance and the dip of the sea-horizon from the top of a mountain $2\frac{3}{4}$ miles high, the earth's mean radius being 3956 miles? [2° 8′ 8″.

8. If the dip of the sea-horizon be 1°, find the height of the mountain, and the distance of the sea-horizon.

9. How far should a coin an inch in diameter be held from the eye to subtend an angle of 1°?

10. Given the earth's equatorial radius, 3962.76 miles, and the angle this radius subtends at the sun, 8″.81 : find the distance of the earth from the sun. [92 780 000 miles nearly.

11. Find the distance across a river, if the base AB be 475 feet; the angle A, 90°; the angle B, 57° 13′ 20″. [737.68 feet.

12. Given CA, 131 feet 5 inches; BC, 109 feet 3 inches; the angle C, 98° 34′ : what is the distance AB? [183 feet.

13. Two ships lying half a mile apart, each observes the angle subtended by the other ship and a fort; the angles are found to be 56° 19′ and 63° 14′ : find the distances of the ships from the fort. [2525, 2710 feet.

14. Given the base AB, $131\frac{1}{2}$ yards; the angle BAD, 50°; the angle BAC, 85° 15′; the angle DBC, 38° 43′; the angle DBA, 94° 13′ : what is the distance CD? Check the work by making two distinct computations from the data. [129.99 yards.

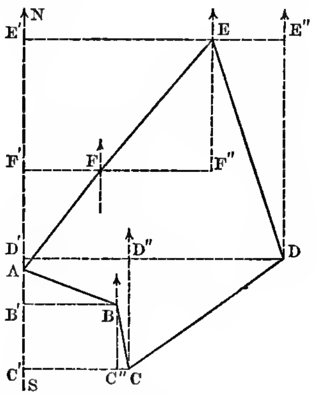
§ 6. COMPASS SURVEYING.

In compass surveying, the *bearing* of a point is the horizontal angle which the line of sight from the observer to the point makes with the north-and-south line through the point of observation. This angle is found by aid of the compass.

The *latitude* of a point is its distance north or south of a given point. The *latitude* of a line is the length of its projection on a north-and-south line; and its *departure* is the length of its projection on an east-and-west line.

E.g. in the figure below, representing a field, the starting point is A, the bearings of the lines AB, BC···, taken in order, are: S. 70° 20′ E. (70° 20′ east from south), S. 10° 15′ E., N. 55° 35′ E., N. 18° 45′ W., S. 40° 55′ W., S. 37° 15′ W.;

and the lengths of these lines, in chains, are: 6.37, 4.28, 12.36, 14.96, 11.15, 8.00.



Through all the points A, B···, are drawn north-and-south lines, marked on the figure with arrows, and east-and-west lines perpendicular to them. The north-and-south line through the starting point A is distinguished as the *meridian*.

The latitude of AB is the length of AB′, the projection of AB on the meridian, and it is computed by multiplying 6.37, the length of AB, by the cosine of 70° 20′, the bearing of AB.

So, the departure of AB is the length of B′B, *i.e.* the product of 6.37 by the sine of 70° 20′.

The latitude of the line BC is the length of BC″, *i.e.* the product of 4.28 by the cosine of 10° 15′, and the departure of BC is the length of C″C, *i.e.* the product of 4.28 by the sine of 10° 15′; and so for the latitudes and departures of the other lines, as shown in the table below.

The *meridian distance* of a point is the distance of the point east or west from the meridian, and the *double meridian distance* of a line is the sum of the meridian distances of its ends.

E.g. the meridian distance of the point B is B′B, and that of C is C′C, which is equal to B′B + C″C.

So, the *double meridian distance* of the line AB is 0 + B′B, and that of BC is B′B + C′C.

When a surveyor has run round a field, *e.g.* that which is described above, and has found and set down the lengths and bearings of the sides, he has next to compute the latitudes and departures of the sides, the meridian distances of the corners, and the double meridian distances of the sides as shown above. He is then ready to compute the areas of certain trapezoids and right triangles, and finally the area of the field; and he takes care to set down his work in such form that it can be easily understood and reviewed, generally in the form of a table as below.

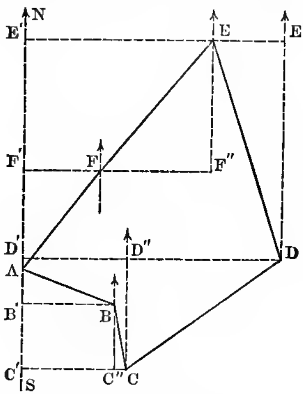
	BEARING.	DISTANCE.	DEP.	M.D.	D.M.D.	LAT.	DOUBLE AREA. +	DOUBLE AREA. −
AB	S. 70° 20′ E.	6.37	5.998	5.998	5.998	−2.144		−12.860
BC	S. 10° 15′ E.	4.28	.761	6.759	12.757	−4.212		−53.732
CD	N. 55° 35′ E.	12.36	10.196	16.955	23.714	6.985	165.642	
DE	N. 18° 45′ W.	14.96	−4.809	12.146	29.101	14.166	412.245	
EF	S. 40° 55′ W.	11.15	−7.303	4.843	16.989	−8.426		−143.149
FA	S. 37° 15′ W.	8.00	−4.843	0.	4.843	−6.369		−30.845
							+577.887	−240.586
							−240.586	
							337.301	

337.301/2 = 168.651 square chains = 16.865 acres.

In this figure there are two right triangles AB′B, FF′A and four trapezoids BB′C′C, CC′D′D, DD′E′E, EE′F′F, so related that the area of the polygon ABCDEF is the excess of the sum of the two trapezoids CC′D′D, DD′E′E over the sum of the two triangles and the other two trapezoids.

$$\begin{aligned} i.e.\ \text{ABCDEF} &= -\text{AB}'\text{B} - \text{BB}'\text{C}'\text{C} + \text{CC}'\text{D}'\text{D} + \text{DD}'\text{E}'\text{E} - \text{EE}'\text{F}'\text{F} - \text{FF}'\text{A} \\ &= \tfrac{1}{2}\,[-\text{AB}'\cdot\text{B}'\text{B} - \text{B}'\text{C}'\cdot(\text{B}'\text{B}+\text{C}'\text{C}) + \text{C}'\text{D}'\cdot(\text{C}'\text{C}+\text{D}'\text{D}) \\ &\qquad + \text{D}'\text{E}'\cdot(\text{D}'\text{D}+\text{E}'\text{E}) - \text{E}'\text{F}'\cdot(\text{E}'\text{E}+\text{F}'\text{F}) - \text{F}'\text{A}\cdot\text{F}'\text{F}] \end{aligned}$$

and it remains only to compute the lines AB′, B′B· · ·, and to add, subtract, and multiply as shown below.



In detail the work may take this form :

1. *To compute the latitudes and departures of the sides:*

S. 70° 20′ E.		S. 10° 15′ E.		N. 55° 35′ E.	
cosine	sine	cosine	sine	cosine	sine
.3365	.9417	.9840	.1779	.5652	.8249
6.37 AB	6.37	4.28 BC	4.28	12.36 CD	12.36
-2.144	5.998	-4.212	.761	6.986	10.196

N. 18° 45′ W.		S. 40° 55′ W.		S. 37° 15′ W.	
cosine	sine	cosine	sine	cosine	sine
.9469½	.3214½	.7556½	.6550	.7960	.6053
14.96 DE	14.96	11.15 EF	11.15	8.00 FA	8.00
14.166	-4.809	-8.426	-7.303	-6.368	-4.842

North latitudes, *northings*, are called positive; south latitudes, *southings*, negative. East departures, *eastings*, are called positive; west departures, *westings*, negative.

2. *To compute the meridian distances:*

B	C	D	E	F	A
5.998	5.998	6.759	16.955	12.146	4.843
	+ .761	+10.196	− 4.809	− 7.303	−4.843
	6.759	16.955	12.146	4.843	0

3. *To compute the double meridian distances:*

AB	BC	CD	DE	EF	FA
5.998	5.998	6.759	16.955	12.146	4.843
	+6.759	+16.955	+12.146	+ 4.843	
	12.757	23.714	29.101	16.989	

4. *To compute the double areas:*

ABB′	BB′C′C	CC′D′D	DD′E′E	EE′F′F	FF′A
5.998	12.757	23.714	29.101	16.989	4.843
× −2.144	× −4.212	× +6.985	× +14.166	× −8.426	× −6.369
−12.860	−53.732	+165.642	+412.245	−143.149	−30.845

QUESTIONS.

1. A surveyor, starting from A, runs N. 22° 37′ E. 3.37 chains to B; thence N. 80° 24′ E. 3.81 chains to C; thence S. 41° 12′ E. 5.29 chains to D; thence S. 62° 45′ W. 6.22½ chains to E: find the latitude and meridian distance of B, C, D, E from A; find the bearing and distance of A from E; find the area of the field ABCDE.

2. Starting at A and chaining along the surface of the ground, a surveyor runs N. 81° 10′ E. 48 chains to B, at an elevation of 4° 15′; thence N. 26° 25′ W. 126 chains to C, at an elevation of 3° 40′; thence S. 73° 50′ W. 45 chains to D, at an elevation of 2° 40′; thence S. 60° E. 85 chains to E, at a depression of 4° 15′: find the horizontal distances AB, BC, CD, DE, and the heights of B, C, D, E above A; find the bearing, distance, and angle of depression, of A from E; find the area of the field ABCDE.

II. GENERAL PROPERTIES OF PLANE ANGLES.

Hitherto the lengths of the sides of a triangle and the magnitudes of the angles have been mainly considered, and little attention has been paid to their directions; but greater generality, as well as greater definiteness, is given to the definitions and theorems of trigonometry if the lines and angles be thought of as directed as well as measured.

Nor is this a new thing: in geography and navigation longitudes are distinguished by the words *east* and *west*, and latitudes by *north* and *south*; a surveyor speaks of his *northings* and *southings* and of his *eastings* and *westings*, and he writes down the bearings of his lines with the significant letters N, S, E, W; in physics the directions and intensities of forces are represented by the directions and lengths of lines.

Even the language is not new: the mathematician merely makes use of the familiar algebraic words *positive* and *negative* as more convenient to him than the commoner words north, south, east, west, up, down, right, left, forward, backward.

§1. DIRECTED LINES.

Hereafter every straight line will be regarded as having not only position but direction also, meaning thereby that a point moving along the line one way will be regarded as moving forward, and a point moving along the line the other way as moving backward. The direction of the line is assumed to be that of forward motion.

If a line represent a force or an actual motion, like that of the winds and the tides, it has a natural direction; otherwise its direction may be assumed at will.

E.g. with a double-track east-and-west railway, the south track may be used habitually by east-bound trains, and the north track by west-bound trains. On the south track a train moves forward when going east, and it goes west only

when backing. On the north track forward motion is westward motion. The two tracks may be regarded as two parallel lines lying close together and having opposite directions.

A *segment* of a line is a limited portion of the line that reaches from one point, the *initial point* of the segment, to another point, the *terminal point.* A segment is a *positive segment* if it reach forward, in the direction of the line, and a *negative segment* if it reach backward.

It is convenient also to speak of the positive and negative ends of a line, meaning by the *positive end* that end which is reached by going forward along the line, from any starting point upon it, and by the *negative end* that end which is reached by going backward.

E.g. if a north-and-south line be directed from south to north, then the north end is the positive end and the south end is the negative end of the line; segments of this line reaching northward are positive segments and segments reaching southward are negative segments.

The direction of a line is indicated by an arrow, or by naming two of its points, the direction being from the point first named towards the other. The direction of a segment is shown by the order of the letters at its extremities, the initial point being named first and the terminal point last.

E.g. the indefinite line OP has its positive direction from O to P, and the segment AB of the line OP is the segment that reaches from the point A to the point B.

If two segments, not necessarily upon the same line, have the same length and be both positive or both negative, they are *equal segments ;* if they have the same length, and be one positive and the other negative, they are *opposite segments.*

ADDITION OF SEGMENTS OF A STRAIGHT LINE.

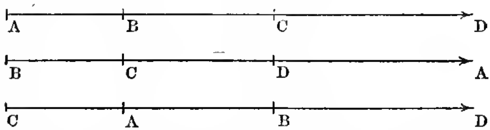
Two or more segments of a straight line are added by placing the initial point of the second segment upon the terminal of the first, the initial point of the third segment

upon the terminal of the second, and so on; and the sum of all the segments so added is the segment that reaches from the first initial to the last terminal point. When a positive segment is added, the terminal point slides forward; when a negative segment is added, it slides backward.

E.g. in the figures below,

$$\mathrm{AB}+\mathrm{BA}=0, \quad \mathrm{AB}+\mathrm{BC}=\mathrm{AC}, \quad \mathrm{AB}+\mathrm{BC}+\mathrm{CA}=0,$$

$$\mathrm{AB}+\mathrm{BC}+\mathrm{CD}=\mathrm{AD}, \quad \mathrm{AB}+\mathrm{BC}+\mathrm{CD}+\mathrm{DA}=0.$$



This addition is analogous to the addition of like numbers, positive and negative, in algebra.

One segment is subtracted from another by adding the opposite of the subtrahend to the minuend, or by placing the initial point of the subtrahend upon that of the minuend; the remainder is then the segment that reaches from the terminal point of the subtrahend to that of the minuend.

QUESTIONS.

1. If from a given starting point one man walk east and another west, each a hundred yards, how far apart are the two men? how far, and in what direction, is the first man from the second? the second man from the first?

2. If the river run five miles an hour, how fast does a boat go, with the current, if the crew can row four miles an hour in still water? against the current?

3. If longitudes alone be under consideration, and west longitudes be marked +, how may east longitudes be marked? how may north and south latitudes be then distinguished?

4. If a traveller go east 50 miles, then west 30 miles, then west 60 miles, then east 20 miles, how far has he gone? and how far, and in what direction, is he from the starting point?

§ 2. DIRECTED PLANES AND ANGLES.

Hereafter every plane will be regarded as having *direction*, meaning thereby that a line swinging about a point in the plane one way will be regarded as swinging forward, and a line swinging the other way as swinging backward. The direction of the plane is that of the forward motion of the line.

If the swinging line has a natural motion like that of the hands of a clock, or a spoke of the fly-wheel of an engine, or an equatorial radius of the earth, then the direction of the plane is determined by this motion; otherwise its direction may be assumed at will.

This swinging motion, as viewed from one side of the plane, is *clockwise, i.e.* left-over-right, and *counter-clockwise, i.e.* right-over-left, as viewed from the other side.

E.g. the apparent daily motion of the sun, as seen by an observer in the northern hemisphere, is clockwise,
and as seen by one in the southern hemisphere it is counter-clockwise;
but to both of them it is the same east-to-west motion, and the plane of the sun's apparent path is an east-to-west plane.
So, the real motion of an equatorial radius of the earth is counter-clockwise if viewed from a point in the northern hemisphere,
and clockwise if from a point in the southern hemisphere;
but it is the same west-to-east motion, and the plane of the equator is a west-to-east plane, whose direction is opposite to that of the sun's apparent path.

An observer to whom forward motion appears counter-clockwise is in *front* of the plane, and looks at its *face;* one to whom forward motion appears clockwise is *back* of the plane.

E.g. the plane of the equator faces northward, and points in the northern hemisphere are in front of it;
but the plane of the sun's apparent path faces southward.

In plane trigonometry the reader always looks at the face of his plane, and to him, therefore, forward motion is always counter-clockwise motion.

DIRECTED ANGLES.

A *plane angle* has been variously defined as "the opening between two lines," as "the inclination of one line to another," as "the difference of direction of two lines," and as "the portion of the plane between the two lines." The words "inclination" and "difference of direction" appear to define the magnitude of the angle rather than the angle itself; but whichever of these definitions be used, it is manifest that an angle may be generated by swinging a line, in the plane of the angle, about the vertex, from one of its bounding lines to the other. The first position of the swinging line is the *initial line*, and the last position is the *terminal line*, of the angle.

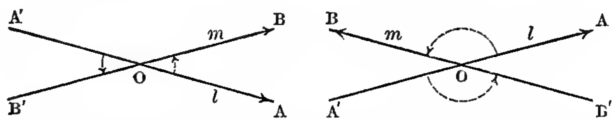
E.g. the minute-hand of a clock generates a right angle every fifteen minutes, and four right angles in an hour.

If the generating line swing forward, in the direction of the plane, it generates a *positive angle;* if it swing backward, it generates a *negative angle.*

Since, in plane trigonometry, the reader always looks at the face of his plane, it follows that positive angles are counter-clockwise angles, and negative angles are clockwise angles.

The angle of two lines is the smaller of the two angles which lie between their positive ends and reaches from the positive end of the line first named to the positive end of the other.

E.g. if the two lines A′A, B′B cross at O, the angle of the two lines A′A, B′B is AOB, and the angle of the two lines B′B, A′A is BOA.



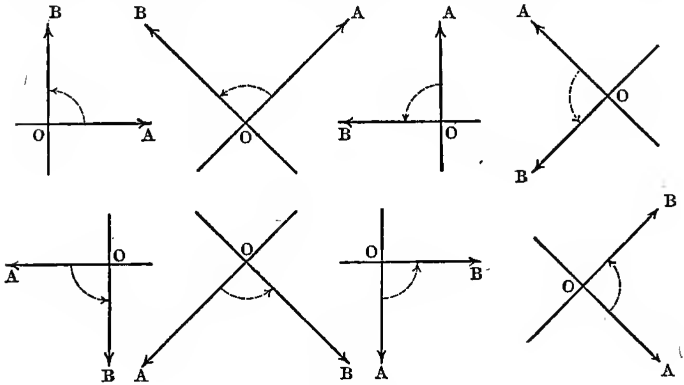
The two bounding lines may be designated by single letters, the initial line being named first.

E.g. if l, m stand for the two lines A′A, B′B, then lm stands for the angle AOB and ml for the angle BOA.

NORMALS.

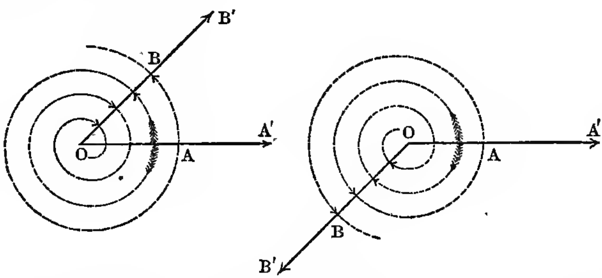
One line is *normal* to another if the first line make a positive right angle with the other.

E.g. in the figures below, OB is normal to OA, but not OA to OB.



EQUAL AND CONGRUENT ANGLES.

If two angles differ by one or more complete revolutions, they are *congruent*; if, when placed one on the other, their initial lines coincide and their terminal lines coincide, they are *equal* or congruent.



E.g. in the figures above all the angles AOB, whether positive or negative, are congruent,

and the angles AOB, A′OB′ are equal, but not AOB, B′OA′.

The smallest angle, positive or negative, of a series of congruent angles is the *primary angle;* and the primary angle is always meant if no other be indicated. It is always smaller than two right angles.

QUESTIONS.

1. If a surveyor by mistake write N. 30° E. 12 chains, instead of N. 30° W. 12 chains, what is his error? and what is the effect, in his map, on the position of every subsequent line and point?

2. If the line a be normal to the line b, what angle does b make with a?

3. Through what angle has the hour-hand of a clock swept from 12 midnight to 12 noon? the minute-hand? the second-hand?

4. If the moon revolve about the earth once in four weeks, what is its angular motion in a year? in a day?

5. How great is the angular motion of the earth upon its own axis in a day? in an hour? in a year?

So, how great is its angular motion in its orbit about the sun in a year? in a day? in a century?

6. What is the angle between a north wind and a northeast wind? a north wind and a southwest wind?

7. If the current carry a chip due south, and the wind carry a feather due east, what is the angle between the arrows that show the directions of the motions of the chip and the feather?

8. If two forces act upon a body, the one vertical and the other horizontal, what is the angle between them? its sign?

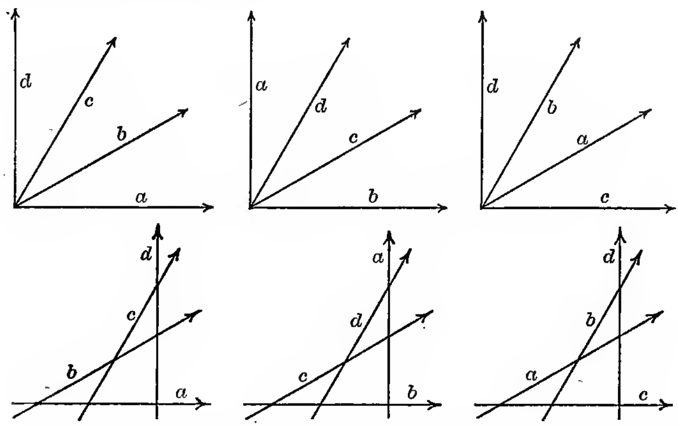
9. If three equal forces acting upon a body be parallel to the three sides of an equilateral triangle, what are the angles between them? Discuss the eight possible cases.

10. If the two hands of a clock start together at noon, what is the angle between them at one o'clock? at two? at three? at six? at nine? at twelve?

ADDITION OF CO-PLANAR ANGLES.

Two or more co-planar angles are added by placing the initial line of the second angle upon the terminal of the first, the initial line of the third angle upon the terminal of the second, and so on; and the sum of all the angles so added is one of the congruent angles reaching from the first initial to the last terminal line. This definition applies whether the vertices of the angles be at the same point or at different points.

When a positive angle is added, the terminal line swings forward; when a negative angle is added, it swings backward. *E.g.* in the figures below,



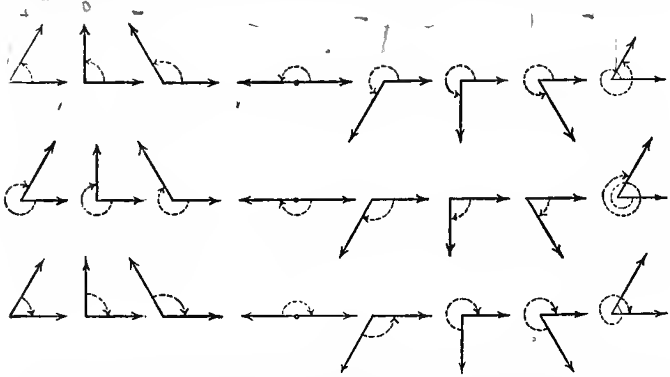
$$ab+ba=0 \text{ (or one of the congruents of 0)}, \quad ab+bc=ac,$$
$$ab+bc+ca=0, \quad ab+bc+cd=ad, \quad ab+bc+cd+da=0.$$

One plane angle is subtracted from another by adding the opposite of the first angle to the other, or by placing the initial line of the first angle upon that of the second; the remainder is then the angle that reaches from the terminal line of the first angle to that of the other.

If the sum of two angles be a positive right angle, either angle is the *complement* of the other; and if their sum be two right angles, either angle is the *supplement* of the other.

QUESTIONS.

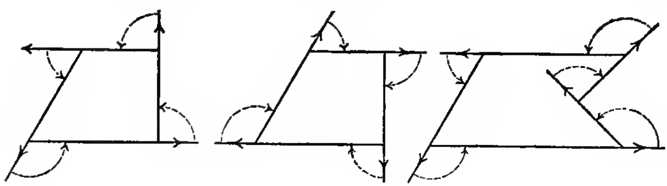
1. Show what angles must be added to the angles below to make the sums positive right angles, and so construct their complements.



2. Show what angles must be added to these angles to make the sums two positive right angles, and so construct their supplements.

3. Show that the angle of two lines equals the angle of any normals to them.

4. If a surveyor, in running round a field, turn at the corners always to the left, what is the sum of the exterior angles of the field ? if he turn always to the right ? if he turn sometimes to the right and sometimes to the left ?



5. If the wind shift from north to northeast, and then from northeast to southeast, through what angle has it shifted ?

§ 3. PROJECTIONS.

The *orthogonal projection* of a point upon a line is the foot of the perpendicular from the point to the line; and, in this book, by *projection* is always meant orthogonal projection. The line on which the projection is made is the *line of projection*, and the perpendicular is the *projecting line*.

The *projection of a segment* of one directed line upon another directed line is the segment of the second line that reaches from the projection of the initial point of the given segment to the projection of its terminal point. The projection is positive if it reach forward in the direction of the line of projection, and negative if it reach backward; its sign may be like or unlike that of the projected segment.

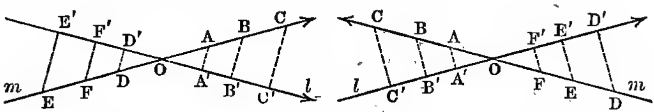
E.g. the shadow of a post on a plane perpendicular to the sun's rays is an orthogonal projection of the post.

Projections upon the same line are *like projections*.

THEOR. 1. *If segments of one directed line be projected upon another such line, the ratios of the projections to the segments are equal.*

Let l, m, be any two directed lines; take AB, CD, EF··· segments of m, and let A′B′, C′D′, E′F′··· be their projections on l;

then will A′B′/AB = C′D′/CD = E′F′/EF···



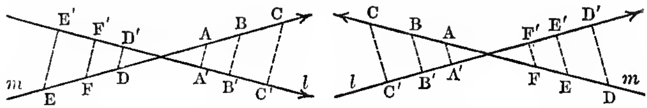
For ∵ the projecting lines AA′, BB′··· are all parallel,

and contrary segments of the same line have contrary projections on another line,

∴ the segments and their projections are proportional;

i.e. A′B′/AB = C′D′/CD = E′F′/EF = OA′/OA = OF′/OF···, both in magnitude and sign. Q.E.D.

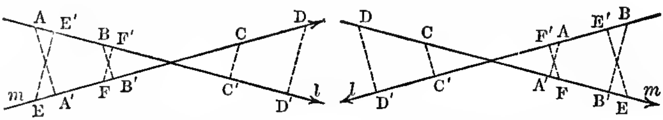
In the first figure the segments AB, EF are positive, and so are their projections A′B′, E′F′, but CD, C′D′ are negative, and all the ratios are positive. In the other figure AB, C′D′, EF are positive, but A′B′, CD, E′F′ are negative, and all the ratios are negative; *i.e.* the ratios are positive if the primary angle



of the two lines be acute, positive or negative; they are negative if the primary angle be obtuse.

COR. 1. *Equal segments of one line have equal projections on another line, and opposite segments have opposite projections.*

COR. 2. *If on each of two directed lines equal segments of the other line be projected, the projections are equal.*



E.g. let l, m be any two lines, AB a segment of l, and CD, EF segments of m equal to AB,

let A′B′ be the projection of AB on m and C′D′, E′F′ the projections of CD, EF on l,

then A′B′, C′D′, E′F′ are equal in magnitude and sign.

In the first figure the segments and their projections are all positive; in the other figure the segments are negative, and their projections are positive.

COR. 3. *If there be two equal angles, and if equal segments of the bounding lines be projected, each upon the other bounding line of its angle, these projections are equal.*

For the two figures may be placed one upon the other, and then cor. 3 becomes a case of cor. 2.

QUESTIONS.

1. A line is 5 feet long and its projection on another line, a, is 4 feet long: how long is its projection on a normal to a?

Can the signs of the projections be found from the data?

2. If a, b be two directed lines at right angles to each other, how long is that segment whose projections on a, b are 5 feet and 12 feet? -5 feet and -12 feet?

Can the sign of the segment be found from the data?

3. Construct lines so that segments of one being projected on the other, the ratios of the projections to the segments shall be 1, $\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, 0; $-\frac{1}{6}$, $-\frac{1}{3}$, $-\frac{2}{3}$, -1, in turn.

4. A pole ten feet long points northward and makes an angle of 45° with the level ground: how long is its shadow, if the sun be directly overhead?

So, how long is its shadow on a north-and-south wall, at sunrise, if the sun rise due east?

Of these two shadows, which is the longer?

So, which is the longer if the inclination be 60°?

From what point of view would the pole appear to be vertical? from what point horizontal?

5. Describe an isosceles triangle by walking due east 100 yards, then northwest 70.7 yards, then southwest 70.7 yards, thus giving direction to the sides.

Project the two sides of this triangle upon the base: what relation have these two projections?

So, project these two sides upon the bisector of the vertical angle: what relation have the two projections now?

6. In an equilateral triangle, whose sides are directed by walking about it and turning to the left at the vertices, how do the projections of the sides upon the base compare in length? in sign?

So, the projections upon a normal to the base?

7. Can a segment of a line be so projected upon another line, that the projection is longer than the segment itself?

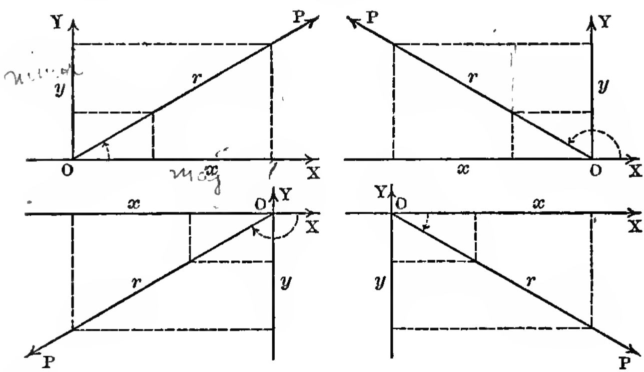
8. Taking note of signs, what is the range of magnitude for the ratio projection/segment? segment/projection?

§ 4. TRIGONOMETRIC RATIOS.

If a segment of the terminal line of an angle be projected upon the initial line and upon a normal to the initial line, the first projection may be called the *major projection* of the segment, the other the *minor projection*, and the ratios of these two projections to the segment and to each other are named as below:

minor projection/segment, the *sine* of the angle,
major projection/segment, the *cosine*,
minor projection/major projection, the *tangent*,
major projection/minor projection, the *cotangent*,
segment/major projection, the *secant*,
segment/minor projection, the *cosecant*.

These definitions apply to all angles whatever their magnitudes or signs, and they include as a special case the definitions given on page 2.



E.g. in the figures above, let XOP be any angle α; let OY be normal to the initial line OX, r any segment of the terminal line OP, x, y the major and minor projections of r;

then $\sin\alpha = y/r$, $\cos\alpha = x/r$, $\tan\alpha = y/x$,
$\csc\alpha = r/y$, $\sec\alpha = r/x$, $\cot\alpha = x/y$.

The segment r may be taken positive or negative; for if the segment be reversed both projections are reversed, and the ratios are unchanged.

Two other functions in common use are the *versed sine* and *coversed sine;* they are defined by the equations

$$\text{vers}\,\alpha = 1 - \cos\alpha, \quad \text{covers}\,\alpha = 1 - \sin\alpha.$$

QUESTIONS.

1. How do the major and minor projections of the segment of a line compare in length with the segment itself? how with each other?

2. Can the sine of an angle be larger than 1? as large as 1? smaller than 1? can the sine be naught? the cosine?

3. Can the tangent of an angle be naught? can it be smaller than 1? as large as 1? larger than 1? how large may the tangent be? the cotangent? the secant? the cosecant?

4. What relations as to sign have a segment and its projections? Draw figures in which:

   all three are positive; all three negative;
   the segment and major projection are positive and the minor projection negative;
   the segment is negative and both projections positive.

5. If two lines be parallel and like directed, what is their angle? How long is the major projection of a segment of one of these lines as to the other? the minor projection?

What are the ratios of this angle?

So, if two parallel lines have opposite directions?

So, if the terminal line be normal to the initial line?

So, if the initial line be normal to the terminal line?

6. Construct the angles $\frac{1}{2}\text{R}, -\frac{1}{2}\text{R}, \quad \frac{3}{2}\text{R}, -\frac{3}{2}\text{R}, \quad \frac{5}{2}\text{R}, -\frac{5}{2}\text{R} \cdots$ and find their ratios. [$\text{R} \equiv$ a right angle.

Which of these angles have the same sines? the same cosines? the same tangents? the same secants?

So, for the angles $\frac{1}{3}\text{R}, -\frac{1}{3}\text{R}, \quad \frac{2}{3}\text{R}, -\frac{2}{3}\text{R}, \quad \frac{4}{3}\text{R}, -\frac{4}{3}\text{R}, \quad \frac{5}{3}\text{R}, -\frac{5}{3}\text{R}$.

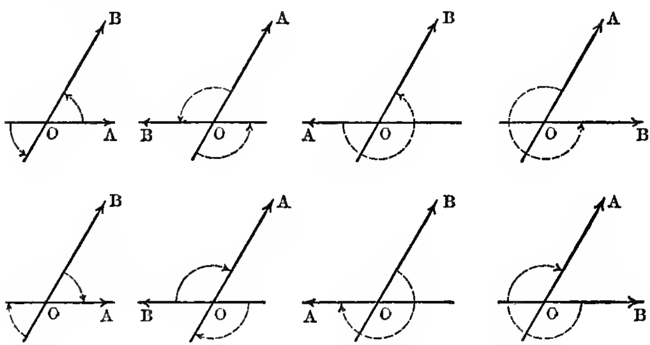
7. Construct $\sin^{-1}\frac{1}{2}, -\frac{2}{3}, \quad \frac{3}{5}, \quad 1, \quad 0; \quad \cos^{-1}\frac{3}{5}, \pm\frac{1}{2}, -1;$
$\tan^{-1}\frac{1}{3}, \quad \frac{3}{4}, \quad 0, -1, \quad \infty; \quad \cot^{-1}\frac{1}{4}, -\frac{3}{4}, \pm 1.$

ANGLES IN THE FOUR QUARTERS.

If there be two lines such that the second line is normal to the first, the plane of these lines is divided into four quarters. The *first quarter* lies between the positive ends of the two lines, the *second quarter* between the positive end of the normal and the negative end of the first line, the *third quarter* between their negative ends, the *fourth quarter* between the negative end of the normal and the positive end of the first line.

An angle is *an angle in the first quarter, in the second quarter, in the third quarter,* or *in the fourth quarter,* according as its terminal line lies in the first, second, third, or fourth quarter, counting from the initial line.

It is therefore an angle in the first quarter if its primary congruent angle be a positive acute angle; in the second quarter, if a positive obtuse angle; in the third quarter, if a negative obtuse angle; in the fourth quarter, if a negative acute angle.



E.g. of the figures above, the first angle and the eighth are angles in the first quarter,
the second and seventh are angles in the second quarter,
the third and sixth are angles in the third quarter,
the fourth and fifth are angles in the fourth quarter.

POSITIVE AND NEGATIVE RATIOS.

THEOR. 2. *The trigonometric ratios of an angle in the first quarter are all positive.*

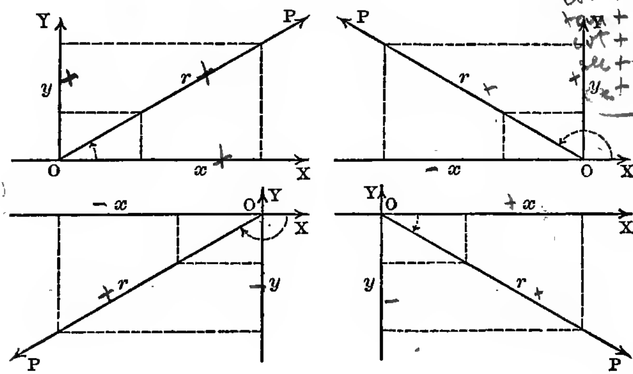
The sine and cosecant of an angle in the second quarter are positive; the cosine, secant, tangent, cotangent are negative.

The tangent and cotangent of an angle in the third quarter are positive; the sine, cosecant, cosine, secant are negative.

The cosine and secant of an angle in the fourth quarter are positive; the sine, cosecant, tangent, cotangent are negative.

For if r be taken positive, and x, y be the major and minor projections of r;

I II
sin + +
cos + −
tan + −
cot + −
sec + −
$\frac{y}{x}$ + +



then $\because$ in the first quarter r, x, y are all positive,
$\therefore$ the ratios y/r, r/y, x/r, r/x, y/x, x/y, are all positive;

and $\because$ in the second quarter r, y are positive, and x negative,
$\therefore$ the ratios y/r, r/y are positive, the rest negative;

and $\because$ in the third quarter r is positive, and x, y negative,
$\therefore$ the ratios y/x, x/y are positive, the rest negative;

and $\because$ in the fourth quarter r, x are positive, and y negative,
$\therefore$ the ratios x/r, r/x are positive, the rest negative.

QUESTIONS.

Show what quarters these angles lie in, and what signs their ratios have : [R ≡ a right angle.

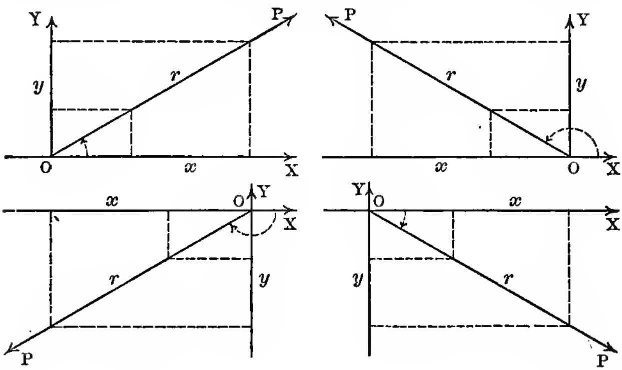
1. $\frac{1}{2}\text{R}, -\frac{1}{2}\text{R},\quad \frac{3}{2}\text{R}, -\frac{3}{2}\text{R},\quad \frac{5}{2}\text{R}, -\frac{5}{2}\text{R},\quad \frac{7}{2}\text{R}, -\frac{7}{2}\text{R} \cdots$
2. $\frac{1}{3}\text{R}, -\frac{1}{3}\text{R},\quad \frac{5}{3}\text{R}, -\frac{5}{3}\text{R},\quad \frac{7}{3}\text{R}, -\frac{7}{3}\text{R},\quad \frac{11}{3}\text{R}, -\frac{11}{3}\text{R} \cdots$
3. $\frac{2}{3}\text{R}, -\frac{2}{3}\text{R},\quad \frac{4}{3}\text{R}, -\frac{4}{3}\text{R},\quad \frac{8}{3}\text{R}, -\frac{8}{3}\text{R},\quad \frac{10}{3}\text{R}, -\frac{10}{3}\text{R} \cdots$
4. $100^\circ,\ 200^\circ,\ 300^\circ,\ 400^\circ,\ 500^\circ,\ 600^\circ,\ 700^\circ,\ 800^\circ,\ 900^\circ.$
5. $^-165^\circ,\ ^-365^\circ,\ ^-565^\circ,\ ^-765^\circ,\ ^-965^\circ,\ ^-1165^\circ,\ ^-1365^\circ.$

Construct the angles below, and find the values of :

6. $\sin\ 225^\circ,\ 585^\circ,\ 810^\circ,\ 960^\circ,\ ^-225^\circ,\ ^-585^\circ,\ ^-960^\circ.$
7. $\cos\ 315^\circ,\ 675^\circ,\ 960^\circ,\ 1110^\circ,\ ^-315^\circ,\ ^-675^\circ,\ ^-1110^\circ.$
8. $\tan\ 495^\circ,\ 945^\circ,\ 1110^\circ,\ 1260^\circ,\ ^-495^\circ,\ ^-915^\circ,\ ^-1260^\circ.$
9. $\cot\ 675^\circ,\ 1035^\circ,\ 1260^\circ,\ 1410^\circ,\ ^-675^\circ,\ ^-1035^\circ,\ ^-1410^\circ.$
10. $\sec\ 855^\circ,\ 1215^\circ,\ 1410^\circ,\ 1560^\circ,\ ^-855^\circ,\ ^-1215^\circ,\ ^-1560^\circ.$
11. $\csc\ 1035^\circ,\ 1395^\circ,\ 1560^\circ,\ 1710^\circ,\ ^-1035^\circ,\ ^-1395^\circ,\ ^-1710^\circ.$

§ 5. RELATIONS OF RATIOS OF A SINGLE ANGLE.

THEOR. 3. *The square of a segment is the sum of the squares of its projections on a line and a normal to the line.*



For these projections are equal to the sides of a right triangle whose hypotenuse is the given segment.

THEOR. 4. *If α be any plane angle, then:*

$$\sin\alpha\cdot\csc\alpha=1,\quad \cos\alpha\cdot\sec\alpha=1,\quad \tan\alpha\cdot\cot\alpha=1;$$
$$\tan\alpha=\sin\alpha/\cos\alpha,\quad \cot\alpha=\cos\alpha/\sin\alpha;$$
$$\sin^2\alpha+\cos^2\alpha=1,\quad \sec^2\alpha=1+\tan^2\alpha,\quad \csc^2\alpha=1+\cot^2\alpha.$$

For let r stand for any segment of the terminal line of the angle, and x, y for its major and minor projections;

then $\because$ $\sin\alpha=y/r$, $\cos\alpha=x/r$, $\tan\alpha=y/x$,
$\csc\alpha=r/y$, $\sec\alpha=r/x$, $\cot\alpha=x/y$, [df.

$\therefore$ $\sin\alpha\cdot\csc\alpha=1$, $\cos\alpha\cdot\sec\alpha=1$, $\tan\alpha\cdot\cot\alpha=1$;

and $\sin\alpha/\cos\alpha=y/r:x/r=y/x=\tan\alpha$,
$\cos\alpha/\sin\alpha=x/r:y/r=x/y=\cot\alpha$.

So, $\because$ $x^2+y^2=r^2$, [theor. 3.

$\therefore$ $x^2/r^2+y^2/r^2=1$, $1+y^2/x^2=r^2/x^2$, $x^2/y^2+1=r^2/y^2$,

i.e. $\cos^2\alpha+\sin^2\alpha=1$, $1+\tan^2\alpha=\sec^2\alpha$, $\cot^2\alpha+1=\csc^2\alpha$.

COR. *If α be any plane angle, then:*

$\sin\alpha=$	$\cos\alpha=$	$\tan\alpha=$	$\cot\alpha=$	$\sec\alpha=$	$\csc\alpha=$
$\sin\alpha$	$\sqrt{(1-\sin^2\alpha)}$	$\dfrac{\sin\alpha}{\sqrt{(1-\sin^2\alpha)}}$	$\dfrac{\sqrt{(1-\sin^2\alpha)}}{\sin\alpha}$	$\dfrac{1}{\sqrt{(1-\sin^2\alpha)}}$	$\dfrac{1}{\sin\alpha}$
$\sqrt{(1-\cos^2\alpha)}$	$\cos\alpha$	$\dfrac{\sqrt{(1-\cos^2\alpha)}}{\cos\alpha}$	$\dfrac{\cos\alpha}{\sqrt{(1-\cos^2\alpha)}}$	$\dfrac{1}{\cos\alpha}$	$\dfrac{1}{\sqrt{(1-\cos^2\alpha)}}$
$\dfrac{\tan\alpha}{\sqrt{(\tan^2\alpha+1)}}$	$\dfrac{1}{\sqrt{(\tan^2\alpha+1)}}$	$\tan\alpha$	$\dfrac{1}{\tan\alpha}$	$\sqrt{(\tan^2\alpha+1)}$	$\dfrac{\sqrt{(\tan^2\alpha+1)}}{\tan\alpha}$
$\dfrac{1}{\sqrt{(\cot^2\alpha+1)}}$	$\dfrac{\cot\alpha}{\sqrt{(\cot^2\alpha+1)}}$	$\dfrac{1}{\cot\alpha}$	$\cot\alpha$	$\dfrac{\sqrt{(\cot^2\alpha+1)}}{\cot\alpha}$	$\sqrt{(\cot^2\alpha+1)}$
$\dfrac{\sqrt{(\sec^2\alpha-1)}}{\sec\alpha}$	$\dfrac{1}{\sec\alpha}$	$\sqrt{(\sec^2\alpha-1)}$	$\dfrac{1}{\sqrt{(\sec^2\alpha-1)}}$	$\sec\alpha$	$\dfrac{\sec\alpha}{\sqrt{(\sec^2\alpha-1)}}$
$\dfrac{1}{\csc\alpha}$	$\dfrac{\sqrt{(\csc^2\alpha-1)}}{\csc\alpha}$	$\dfrac{1}{\sqrt{(\csc^2\alpha-1)}}$	$\sqrt{(\csc^2\alpha-1)}$	$\dfrac{\csc\alpha}{\sqrt{(\csc^2\alpha-1)}}$	$\csc\alpha$

The proof of these equations is left as an exercise for the pupil, but certain relations may be noted:

The values of the cosecant set down in the sixth column are reciprocals of those of the sine in the first;

those of the secant in the fifth column of those of the cosine in the second,

and those of the cotangent in the fourth column of those of the tangent in the third.

The values of the tangent and cotangent set down in the third and fourth columns are quotients of the values of the sine and cosine in the first and second columns.

QUESTIONS.

1. For a given value of the sine, how many values has the cosecant? the cosine? the secant? the tangent? the cotangent?

What signs have the radicals in each of the four quarters?

2. For a given value of the cosecant, how many values has each of the other five ratios?

3. So, for a given value of the cosine? of the secant? of the tangent? of the cotangent?

4. Construct the two angles whose sines are $+\frac{3}{5}$, and show that the two cosines are $+\frac{4}{5}$ and $-\frac{4}{5}$.

5. Construct the two angles whose cosines are $-\frac{4}{5}$, and show that the two sines are $+\frac{3}{5}$ and $-\frac{3}{5}$.

6. Construct the two angles whose tangents are $+\frac{3}{4}$, and thence show the double values of the sine, the cosecant, the cosine, the secant, and the single value of the cotangent.

7. Show that the formulæ of the corollary to theor. 4, taken two and two, are symmetric:

those for sine, in terms of cosine, tangent, secant, $\cdots$
with those for cosine, in terms of sine, cotangent, $\cdots$;

those for tangent, in terms of sine, cosine, secant, $\cdots$
with those for cotangent, in terms of cosine, sine, $\cdots$;

those for secant, in terms of sine, cosine, tangent, $\cdots$
with those for cosecant, in terms of cosine, sine, $\cdots$.

8. Show that the formulæ proved in examples 3, 4, page 7, hold true with the broader definitions of the trigonometric ratios, given on page 34.

9. Show that the methods of proof shown in examples 5, 6, page 7, apply to the formulæ in the corollary to theor. 4.

§ 6. RATIOS OF RELATED ANGLES.

THE RATIOS OF OPPOSITE ANGLES.

THEOR. 5. *If α be any plane angle, then:*

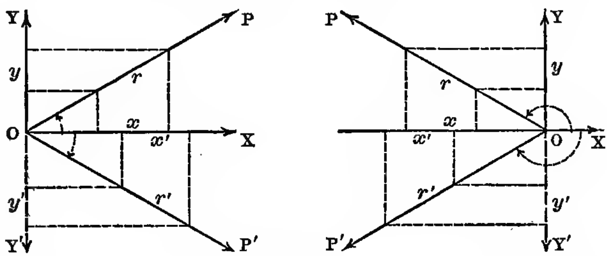
$$\sin(-\alpha) = -\sin\alpha, \quad \cos(-\alpha) = +\cos\alpha,$$
$$\tan(-\alpha) = -\tan\alpha, \quad \cot(-\alpha) = -\cot\alpha,$$
$$\sec(-\alpha) = +\sec\alpha, \quad \csc(-\alpha) = -\csc\alpha.$$

sin = - si
cos = co
tan = - ta
cot = - c
sec = s
csc = - c

For, let XOP, XOP′ be any opposite angles α, $-\alpha$, having the same vertex O, the same initial line OX, and the terminal lines OP, OP′ symmetric as to OX;

draw OY normal to OX and OY′ opposite to OY.

On OP, OP′ take equal segments r, r' and let their major and minor projections, *i.e.* their projections on OX, OY, be $x, y, \quad x', y'$;



then ∵ the angles XOP, P′OX are equal, and so are the segments r, r', [constr.

∴ the projections of r, r' on OX are equal. [theor. 1, cor. 3.

So, ∵ the angles POY, Y′OP′ are equal,

∴ the projection of r on OY equals the projection of r' on OY′, and is the opposite of the projection of r' on OY; [theor. 1, cor. 1.

i.e. $\quad r = r', \quad x = x', \quad y = -y'$,

$\therefore \sin(-\alpha), \equiv y'/r' = -y/r, \equiv -\sin\alpha,$

$\cos(-\alpha), \equiv x'/r' = x/r, \equiv \cos\alpha;$

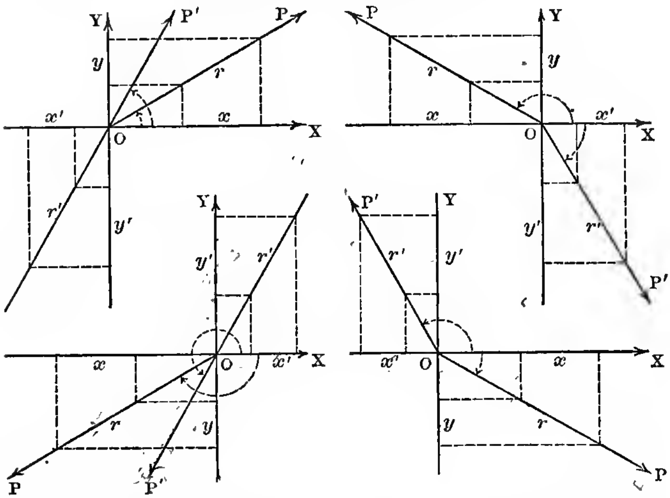
and so for the rest.

THE RATIOS OF THE COMPLEMENT OF AN ANGLE.

THEOR. 6. *If α be any plane angle, then:*

$$\sin co\text{-}\alpha = \cos \alpha, \quad \cos co\text{-}\alpha = \sin \alpha, \quad \tan co\text{-}\alpha = \cot \alpha,$$
$$\csc co\text{-}\alpha = \sec \alpha, \quad \sec co\text{-}\alpha = \csc \alpha, \quad \cot co\text{-}\alpha = \tan \alpha.$$

For let XOP be any angle α; draw OY normal to OX, and OP' making the angle P'OY equal to XOP;



then ∵ XOP' + P'OY = R, XOP + POY = R, P'OY = XOP, [constr.

∴ XOP' = co-α = POY.

On OP, OP' take r, r' equal segments, and let their major and minor projections, *i.e.* their projections on OX, OY, be x, y, x', y';

then ∵ XOP, P'OY are equal angles, and so are XOP', POY,

∴ the major projection of r equals the minor projection of r',

and the minor projection of r equals the major projection of r';

i.e. $r = r'$, $x = y'$, $y = x'$,

∴ sin *co*-α, $\equiv y'/r' = x/r$, $\equiv$ cos α; and so for the rest.

right ∠

THE RATIOS OF R + α.

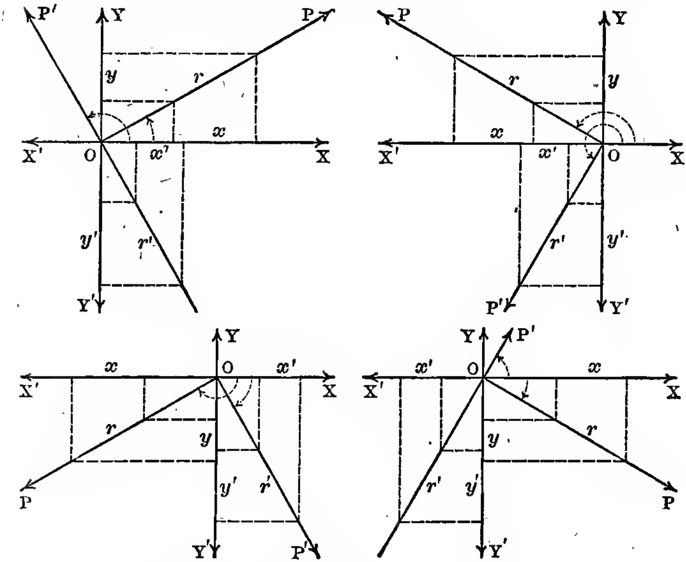
THEOR. 7. *If α be any plane angle, and* R *a right angle, then:*

$$\sin(\mathrm{R}+\alpha)=\cos\alpha, \qquad \cos(\mathrm{R}+\alpha)=-\sin\alpha,$$
$$\tan(\mathrm{R}+\alpha)=-\cot\alpha, \qquad \cot(\mathrm{R}+\alpha)=-\tan\alpha,$$
$$\sec(\mathrm{R}+\alpha)=-\csc\alpha, \qquad \csc(\mathrm{R}+\alpha)=\sec\alpha.$$

sin =
cos =
tan =
cot =
sec =
csc

Let XOP be any angle α, draw OY normal to OX, OP′ normal to OP, and OX′ opposite to OX; then XOP′ = R + α.

On OP, OP′ take equal segments r, r', and let their major and minor projections be x, y, x', y';



then ∵ XOP, YOP′ are equal angles,

∴ the projection of r on OX equals that of r' on OY;

and ∵ POY, P′OX′ are equal angles,

∴ the projection of r on OY equals that of r' on OX′, and is the opposite of the projection of r' on OX;

i.e. $r=r'$, $x=y'$, $y=-x'$;

∴ $\sin(\mathrm{R}+\alpha), \equiv y'/r'=x/r, \equiv \cos\alpha$; and so for the rest.

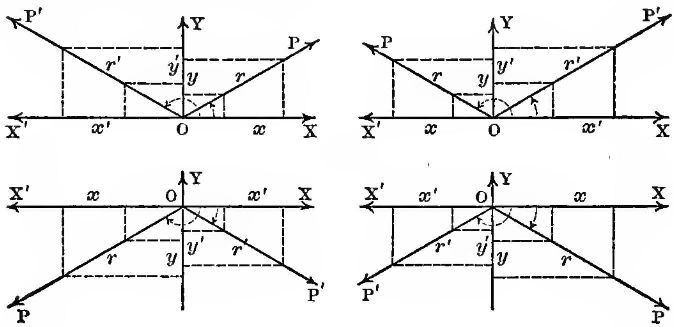
THE RATIOS OF THE SUPPLEMENT OF AN ANGLE.

THEOR. 8. *If α be any plane angle, then:*

$$\sin \sup \alpha = \sin \alpha, \qquad \cos \sup \alpha = -\cos \alpha,$$
$$\tan \sup \alpha = -\tan \alpha, \qquad \cot \sup \alpha = -\cot \alpha,$$
$$\sec \sup \alpha = -\sec \alpha, \qquad \csc \sup \alpha = \csc \alpha.$$

Let XOP be any plane angle α, draw OY normal to OX, and OX′ opposite to OX;

draw OP′, making the angle P′OX′ equal to α;



then ∵ XOP′ + P′OX′ = 2R, and XOP = P′OX′, [constr.

∴ XOP, XOP′ are supplementary angles.

On OP, OP′, take equal segments r, r', and let their major and minor projections, *i.e.* their projections on OX, OY, be $x, y,\ \ x', y'$;

then ∵ XOP, P′OX′ are equal angles,

∴ the projection of r on OX equals that of r' on OX′, and is the opposite of the projection of r' on OX;

and ∵ POY, YOP′ are equal angles,

∴ the projections of r, r' on OY are equal;

i.e. $r = r'$, $x = -x'$, $y = y'$;

∴ $\sin \sup \alpha, \equiv y'/r' = y/r, \equiv \sin \alpha,$

$\cos \sup \alpha, \equiv x'/r' = -x/r, \equiv -\cos \alpha;$

and so for the rest. Q.E.D.

THE RATIOS OF $2R+\alpha$.

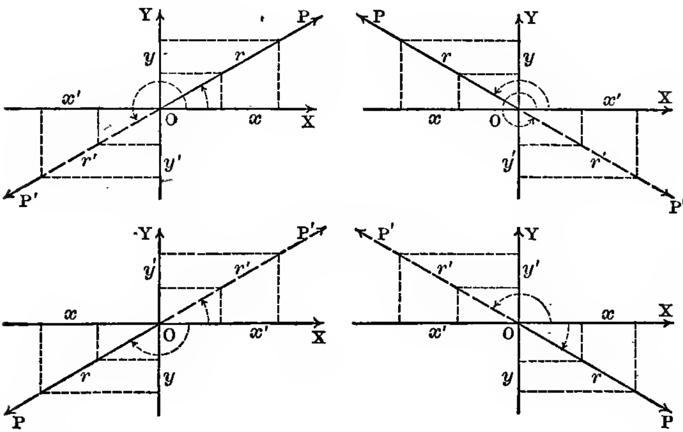
THEOR. 9. *If α be any plane angle, and* R *a right angle, then:*

$\sin(2R+\alpha) = -\sin\alpha$, $\quad \cos(2R+\alpha) = -\cos\alpha$, sin = − cos = −

$\tan(2R+\alpha) = \tan\alpha$, $\quad \cot(2R+\alpha) = \cot\alpha$, tan = ta cot = co

$\sec(2R+\alpha) = -\sec\alpha$, $\quad \csc(2R+\alpha) = -\csc\alpha$. sec = − csc = −

Let XOP be any plane angle α; draw OY normal to OX, OP′ opposite to OP;



then ∵ $POP' = 2R$,

∴ $XOP' = 2R+\alpha$.

On OP, OP′ take equal segments r, r', and let their major and minor projections, *i.e.* their projections on OX, OY, be x, y, x', y';

then ∵ r, r' are opposite segments of OP,

∴ their major projections are opposite, and so are their minor projections;

i.e. $\quad r = r', \quad x = -x', \quad y = -y'$;

∴ $\sin(2R+\alpha), \equiv y'/r' = -y/r, \equiv -\sin\alpha$,

$\cos(2R+\alpha), \equiv x'/r' = -x/r, \equiv -\cos\alpha$;

and so for the rest. Q. E. D.

QUESTIONS.

1. Given the ratios of $\frac{1}{2}$R: by aid of theors. 5–9, find the ratios of $\frac{3}{2}$R, $\frac{5}{2}$R, $\frac{7}{2}$R$\cdots$, and of $-\frac{1}{2}$R, $-\frac{3}{2}$R, $-\frac{5}{2}$R, $-\frac{7}{2}$R $\cdots$.

2. Given the ratios of $\frac{1}{3}$R: find those of $\frac{2}{3}$R, $\frac{4}{3}$R, $\frac{5}{3}$R $\cdots$, and of $-\frac{1}{3}$R, $-\frac{2}{3}$R, $-\frac{4}{3}$R, $-\frac{5}{3}$R $\cdots$.

3. Given the ratios of R: find those of 0, 2R, 3R, 4R $\cdots$, and of $-$R, -2R, -3R, -4R $\cdots$.

4. Find the ratios of R$+\alpha$, as the complement of $-\alpha$.

5. Find the ratios of 2R$+\alpha$, as the supplement of $-\alpha$.

6. Find the ratios of $\alpha-$R, as the opposite of co-α.

7. Find the ratios of 3R$+\alpha$, and of 3R$-\alpha$.

8. Find the ratios of 2R$-\alpha$, as the complement of $\alpha-$R.

9. Find the ratios of 4R$+\alpha$, as supplement of $-$(2R$+\alpha$).

10. Given $\cos\alpha=\frac{1}{2}$: find $\sin^{-1}\frac{1}{2}$, $\sin^{-1}-\frac{1}{2}$.

11. Given $\csc\alpha=2$: find $\sec^{-1}2$, $\sec^{-1}-2$.

12. What angles have the same sine as α? the same cosine? the same tangent? the same secant? the same cosecant?

In ratios of positive angles less than R, express the values of:

13. sin 135°, 335°, $-535°$, $-735°$, $\frac{27}{5}$R, $-\frac{29}{7}$R, $\frac{31}{9}$R, $-\frac{33}{11}$R.

14. cos 235°, 435°, $-635°$, $-835°$, $\frac{29}{5}$R, $-\frac{31}{7}$R, $\frac{33}{9}$R, $-\frac{35}{11}$R.

15. tan 335°, 535°, $-735°$, $-935°$, $\frac{31}{5}$R, $-\frac{33}{7}$R, $\frac{35}{9}$R, $-\frac{37}{11}$R.

16. cot 435°, 635°, $-835°$, $-1035°$, $\frac{33}{5}$R, $-\frac{35}{7}$R, $\frac{37}{9}$R, $-\frac{39}{11}$R.

17. sec 535°, 735°, $-935°$, $-1135°$, $\frac{35}{5}$R, $-\frac{37}{7}$R, $\frac{39}{9}$R, $-\frac{41}{11}$R.

18. csc 635°, 835°, $-1035°$, $-1235°$, $\frac{37}{5}$R, $-\frac{39}{7}$R, $\frac{41}{9}$R, $-\frac{43}{11}$R.

In ratios of positive angles not greater than $\frac{1}{2}$R, express the values of:

19. sin 50°, 150°, $-250°$, $-350°$, $\frac{3}{12}$R, -4R.

20. cos 60°, 160°, $-260°$, $-360°$, $\frac{5}{12}$R, $-\frac{14}{3}$R.

21. tan 70°, 170°, $-270°$, $-370°$, $\frac{7}{12}$R, $-\frac{16}{3}$R.

22. cot 80°, 180°, $-280°$, $-380°$, $\frac{9}{12}$R, -6R.

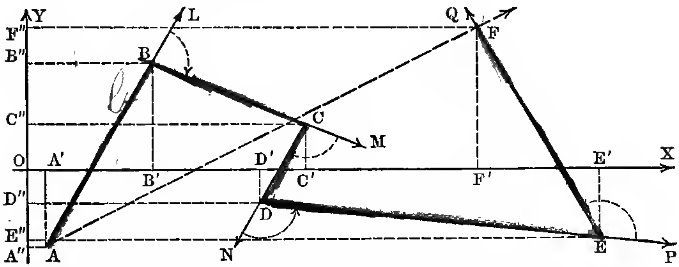
23. sec 90°, 190°, $-290°$, $-390°$, $\frac{11}{12}$R, $-\frac{20}{3}$R.

sec 10 − csc 20

24. csc 100°, 200°, $-300°$, $-400°$, $\frac{13}{12}$R, $-\frac{22}{3}$R.

§ 7. PROJECTION OF A BROKEN LINE.

The *projection of a broken line* upon a straight line is the sum of the projections upon it of the segments that constitute the broken line, and it is identical with the like projection of the single segment that reaches from the initial to the terminal point of the broken line.



THEOR. 10. *The major projection of a segment of a line is equal to the product of the segment by the cosine of the angle the line makes with the line of projection; and the minor projection is equal to the product of the segment by the sine of this angle.* [df. sine, cosine.

COR. *The major projection of a broken line is the sum of the products of the segments each by the cosine of the angle its line makes with the line of projection, and the minor projection is the sum of their products by the sines of these angles.*

E.g. in the figure above, let the broken line ABCDEF be formed by the segments AB, BC, CD ⋯, of the lines AL, BM, CN ⋯,

let $\alpha, \beta, \gamma \cdots \phi$ stand for the angles OX-AL, OX-BM, OX-CN ⋯ OX-AF.

then maj-proj ABCDEF $= A'B' + B'C' + C'D' \cdots$
$= AB \cos\alpha + BC \cos\beta + CD \cos\gamma \cdots$
$= A'F' = AF \cos\phi,$

and min-proj ABCDEF $= A''B'' + B''C'' + C''D'' \cdots$
$= AB \sin\alpha + BC \sin\beta + CD \sin\gamma \cdots$
$= A''F'' = AF \sin\phi.$

§ 8. RATIOS OF THE SUM, AND OF THE DIFFERENCE, OF TWO ANGLES.

THEOR. 11. *If α, β be any two plane angles, then:*

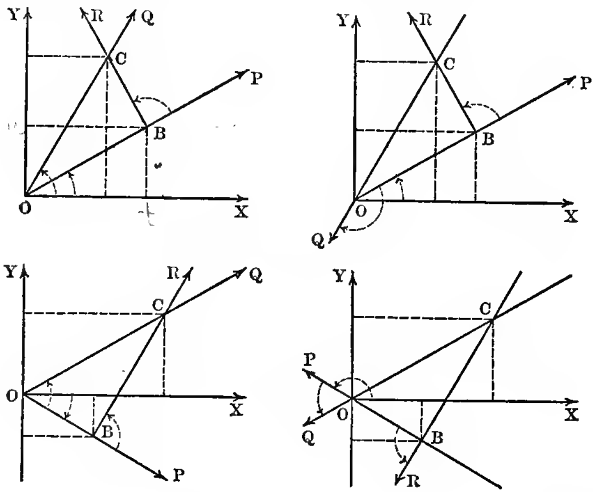
$$\left.\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta,\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta,\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta,\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta. \end{aligned}\right\}$$

learn

For, let XOP, POQ be any two plane angles α, β, so placed that their vertices coincide, and the terminal line, OP, of α, is the initial line of β;

then XOQ $= \alpha+\beta$.

On OQ take any segment OC, and draw CR normal to OP at B;



then ∵ the major projection of OC, as to OX, equals the like projection of the broken line OBC,

i.e. maj-proj OC = maj-proj OB + maj-proj BC, [df.

∴ maj-proj OC/OC = maj-proj OB/OC + maj-proj BC/OC

= maj-proj OB/OB · OB/OC + maj-proj BC/BC · BC/OC.

But $\quad$ maj-proj OC/OC $=\cos$ XOQ $=\cos(\alpha+\beta)$, [df.
maj-proj OB/OB $=\cos$ XOP $=\cos\alpha$;

and ∵ OB, BC are the major and minor projections of OC, as to OP,

∴ OB/OC $=\cos$ POQ $=\cos\beta$,
BC/OC $=\sin$ POQ $=\sin\beta$;

and ∵ angle OX-BR $=$ XOP $+$ PBR $=\alpha+$ R,

∴ maj-proj BC/BC $=\cos$ OX-BR $=\cos(\alpha+\text{R})=-\sin\alpha$, [th. 7.

∴ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. Q. E. D.

And ∵ α, β may be any plane angles positive, or negative,

and $\quad \alpha-\beta=\alpha+(-\beta)$ whatever the sign or magnitude of β,

∴ $\cos(\alpha-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$ [df.
$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$. Q. E. D. [theor. 5.

So, ∵ the minor projection of OC equals the like projection of the broken line OBC,

∴ min-proj OC/OC $=$ min-proj OB/OC $+$ min-proj BC/OC
$=$ min-proj OB/OB $\cdot$ OB/OC $+$ min-proj BC/BC $\cdot$ BC/OC,

∴ $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin(\text{R}+\alpha)\sin\beta$
$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, Q. E. D.

and $\quad \sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$,
$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$. Q. E. D.

COR. 1.
$$\left|\begin{array}{l} \tan(\alpha+\beta)=(\tan\alpha+\tan\beta)/(1-\tan\alpha\tan\beta), \\ \tan(\alpha-\beta)=(\tan\alpha-\tan\beta)/(1+\tan\alpha\tan\beta). \end{array}\right\}$$

For $\quad \tan(\alpha+\beta)=\sin(\alpha+\beta)/\cos(\alpha+\beta)$ [theor. 4.
$=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)/(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$.

Divide both terms of this fraction by $\cos\alpha\cos\beta$;

then $\quad \tan(\alpha+\beta)=(\tan\alpha+\tan\beta)/(1-\tan\alpha\tan\beta)$;

and so for $\tan(\alpha-\beta)$. Q. E. D.

COR. 2.
$$\left|\begin{array}{l} \sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta, \\ \sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta, \\ \cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta, \\ \cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta. \end{array}\right\}$$

CONVERSION FORMULÆ.

THEOR. 12. *If α, β be any two plane angles, then:*

$$\sin\alpha+\sin\beta=2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$
$$\sin\alpha-\sin\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta),$$
$$\cos\alpha+\cos\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$
$$\cos\alpha-\cos\beta=-2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$

For let γ, δ be two plane angles such that

$$\gamma=\tfrac{1}{2}(\alpha+\beta) \quad\text{and}\quad \delta=\tfrac{1}{2}(\alpha-\beta),$$

then, $\gamma+\delta=\alpha$, and $\gamma-\delta=\beta$,

and $\because \sin(\gamma+\delta)+\sin(\gamma-\delta)=2\sin\gamma\cos\delta$, [theor. 11, cor. 2.

$\therefore \sin\alpha+\sin\beta=2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$;

and so for the other formulæ. Q.E.D.

QUESTIONS.

1. Given $\sin\alpha=.3$, $\sin\beta=.6$: find $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, each correct to three decimal places.

2. From the sine and cosine of $30°$ and $45°$, find the ratios of $15°$ and $75°$, then those of $105°$, $165°$, $195°$, $255°$, $285°$, $345°$.

3. Remembering the ratios of 0, R, 2R $\cdots$, verify theors. 5–9, by aid of theor. 11. What is the defect in this proof?

4. If α, β be any plane angles, then

$$\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha,$$
$$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha.$$

5. Divide the values of $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, each in turn by $\cos\alpha\cos\beta$, $\sin\alpha\sin\beta$, $\sin\alpha\cos\beta$, $\cos\alpha\sin\beta$, and express the results in terms of $\tan\alpha$, $\tan\beta$.

6. Show that, with α, β each smaller than two right angles, there may be thirty-two distinct figures to illustrate theor. 11, each differing from the rest in some important particular.

E.g. α, β, $\alpha+\beta$ may all be positive acute angles, or α, β may be positive acute angles, and $\alpha+\beta$ an obtuse angle.

7. If $\sin\alpha=\sin\beta$ and $\cos\alpha=\cos\beta$ prove that $\cos(\alpha-\beta)=1$, and so that α, β are either equal or congruent.

8. Prove that $\cos\alpha+\cos(120°+\alpha)+\cos(120°-\alpha)=0$.

9. Prove that $\sin^2 10° - \cos^2 190° = \cos 200°$.

10. If A, B, C, D be any four plane angles, then
$$\sin(A-B)\sin(C-D)+\sin(B-C)\sin(A-D)$$
$$+\sin(C-A)\sin(B-D)=0.$$

11. Let $\alpha, \beta, \gamma, \cdots \lambda$ be any plane angles, then
$$\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)+\cdots$$
$$+\cos(\lambda+\alpha)\sin(\lambda-\alpha)=0.$$

12. Solve the equation $\cos 3\alpha+\cos 2\alpha+\cos\alpha=0$.
[$3\alpha=2\alpha+\alpha,\ \alpha=2\alpha-\alpha$.

13. Prove the identity [$7\alpha=4\alpha+3\alpha,\ 5\alpha=4\alpha+\alpha\cdots$
$$\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha=4\sin 4\alpha\cos 2\alpha\cos\alpha.$$

14. Given $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{1}{3}$: find $\tan(\alpha+\beta)$.

15. Given $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{1}{5}$, $\tan\gamma=\frac{1}{8}$: find $\tan(\alpha+\overline{\beta+\gamma})$.

16. Show that
$$\sin 28°+\sin 14°=2\sin 21°\cos 7°,$$
$$\sin 28°-\sin 14°=2\cos 21°\sin 7°,$$
$$\cos 28°+\cos 14°=2\cos 21°\cos 7°,$$
$$\cos 28°-\cos 14°=-2\sin 21°\sin 7°,$$
$$\sin 80°-\sin 20°=\cos 50°,$$
$$\sin 75°-\sin 45°=\sin 15°.$$

17. In terms of tangents and cotangents find the values of:
$$(\sin\alpha+\sin\beta)/(\cos\alpha+\cos\beta),$$
$$(\sin\alpha-\sin\beta)/(\cos\alpha+\cos\beta),$$
$$(\sin\alpha+\sin\beta)/(\cos\alpha-\cos\beta),$$
$$(\sin\alpha-\sin\beta)/(\cos\alpha-\cos\beta),$$
$$(\sin\alpha+\sin\beta)/(\sin\alpha-\sin\beta),$$
$$(\cos\alpha+\cos\beta)/(\cos\alpha-\cos\beta).$$

18. Given $\alpha=60°$, $\beta=45°$: find $\tan 52°\,30'$, $\tan 7°\,30'$. [th. 12.
So, given $\alpha=45°$, $\beta=30°$: find $\tan 37°\,30'$, $\tan 7°\,30'$.

19. Given $\sin 15°=.25882$, $\sin 45°=\sqrt{\frac{1}{2}}$: find $\cos 60°$, $\cos 30°$.

20. Given $\cos 75°=.25882$: find $\sin 30°$.

21. Given $\cos 17°=.9563$, $\sin 23°=.3907$: find $\tan 7°$, $\tan 40°$, $\sin 20°$, $\cos 3°\,30'$.

§9. RATIOS OF DOUBLE ANGLES AND OF HALF ANGLES.

THEOR. 13. *If α be any plane angle, then:*

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\tan\alpha/(1+\tan^2\alpha),$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= 2\cos^2\alpha - 1$$
$$= 1 - 2\sin^2\alpha$$
$$= (1-\tan^2\alpha)/(1+\tan^2\alpha),$$
$$\tan 2\alpha = 2\tan\alpha/(1-\tan^2\alpha).$$

For $\sin 2\alpha = \sin(\alpha+\alpha)$

$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$ [theor. 11.

$= 2\sin\alpha\cos\alpha$; Q.E.D.

$= (2\sin\alpha/\cos\alpha)\cdot\cos^2\alpha$

$= 2\tan\alpha/\sec^2\alpha$

$= 2\tan\alpha/(1+\tan^2\alpha)$. [theor. 4.

So, $\cos 2\alpha = \cos(\alpha+\alpha)$

$= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$ [theor. 11.

$= \cos^2\alpha - \sin^2\alpha$; Q.E.D.

and $\cos 2\alpha = \cos^2\alpha - (1-\cos^2\alpha)$

$= 2\cos^2\alpha - 1$; Q.E.D. [theor. 4.

and $\cos 2\alpha = (1-\sin^2\alpha) - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$; Q.E.D.

and $\cos 2\alpha = (\cos^2\alpha/\cos^2\alpha - \sin^2\alpha/\cos^2\alpha)\cdot\cos^2\alpha$

$= (1-\tan^2\alpha)/\sec^2\alpha$

$= (1-\tan^2\alpha)/(1+\tan^2\alpha)$. Q.E.D. [theor. 4.

So, $\tan 2\alpha = \tan(\alpha+\alpha)$

$= (\tan\alpha+\tan\alpha)/(1-\tan\alpha\tan\alpha)$, [th. 11, cor. 1.

$= 2\tan\alpha/(1-\tan^2\alpha)$. Q.E.D.

COR. $\sin\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha,$

$\cos\alpha = 2\cos^2\tfrac{1}{2}\alpha - 1 = 1 - 2\sin^2\tfrac{1}{2}\alpha,$ [theor. 4.

$1+\cos\alpha = 2\cos^2\tfrac{1}{2}\alpha,$

$1-\cos\alpha = 2\sin^2\tfrac{1}{2}\alpha.$

THEOR. 14. *If α be any plane angle, then:*

$$\sin\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1-\cos\alpha)},$$
$$\cos\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1+\cos\alpha)},$$
$$\tan\tfrac{1}{2}\alpha = \sin\alpha/(1+\cos\alpha)$$
$$= (1-\cos\alpha)/\sin\alpha$$
$$= \sqrt{[(1-\cos\alpha)/(1+\cos\alpha)]}.$$

For $\because 2\sin^2\tfrac{1}{2}\alpha = 1-\cos\alpha$, [theor. 13, cor.

$\therefore \sin\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1-\cos\alpha)}$. Q.E.D.

So, $\because 2\cos^2\tfrac{1}{2}\alpha = 1+\cos\alpha$, [theor. 13, cor.

$\therefore \cos\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1+\cos\alpha)}$. Q.E.D.

So, $\because \tan\tfrac{1}{2}\alpha = \sin\tfrac{1}{2}\alpha/\cos\tfrac{1}{2}\alpha$, [theor. 4.

$\therefore \tan\tfrac{1}{2}\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha/2\cos^2\tfrac{1}{2}\alpha$

$= \sin\alpha/(1+\cos\alpha)$; Q.E.D. [theor. 13, cor.

and $\tan\tfrac{1}{2}\alpha = 2\sin^2\tfrac{1}{2}\alpha/2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$

$= (1-\cos\alpha)/\sin\alpha$. Q.E.D.

So, $\tan\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1-\cos\alpha)}/\sqrt{\tfrac{1}{2}(1+\cos\alpha)}$

$= \sqrt{[(1-\cos\alpha)/(1+\cos\alpha)]}$. Q.E.D.

QUESTIONS.

1. From the known value of cos 30°, find the ratios of 15°; from cos 15° find the ratios of 7° 30′; from cos 7° 30′ find the ratios of 3° 45′, and so on, each correct to four decimal places.

If $A+B+C=2R$, then:

2. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

3. $\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$.

4. $\cos A + \cos B + \cos C = 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C + 1$.

Prove the identities:

5. $\csc 2\alpha + \cot 2\alpha = \cot\alpha$; $\cos\alpha = \cos^4\tfrac{1}{2}\alpha - \sin^4\tfrac{1}{2}\alpha$.

6. $\tan\alpha + \cot\alpha = 2\csc 2\alpha$; $\tan\alpha - \cot\alpha = -2\cot 2\alpha$.

7. $\tan(\tfrac{1}{2}R - \tfrac{1}{2}\alpha) + \cot(\tfrac{1}{2}R - \tfrac{1}{2}\alpha) = 2\sec\alpha$.

8. $(\cos\alpha + \sin\alpha)/(\cos\alpha - \sin\alpha) = \tan 2\alpha + \sec 2\alpha$.

9. $\tan^2(\tfrac{1}{2}R + \tfrac{1}{2}\alpha) = (\sec\alpha + \tan\alpha)/(\sec\alpha - \tan\alpha)$.

§ 10. RATIOS OF THE SUM OF THREE OR MORE ANGLES, AND OF MULTIPLE ANGLES.

THEOR. 15. *If α, β, γ be any three plane angles, then:*

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$$
$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma$$
$$=\cos\alpha\cos\beta\cos\gamma\,(\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma).$$

$$\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma$$
$$-\cos\beta\sin\gamma\sin\alpha-\cos\gamma\sin\alpha\sin\beta$$
$$=\cos\alpha\cos\beta\cos\gamma\,(1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta).$$

$$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta}.$$

Prove by expanding $\sin(\overline{\alpha+\beta}+\gamma)$, $\cos(\overline{\alpha+\beta}+\gamma)$. [th. 11.

COR. 1. *If α, β, γ, $\cdots$ be any plane angles, then:*

$$\sin(\alpha+\beta+\gamma\cdots)/\cos\alpha\cos\beta\cos\gamma\cdots$$
$$=\Sigma\tan\alpha-\Sigma\tan\alpha\tan\beta\tan\gamma+\cdots,$$
$$\cos(\alpha+\beta+\gamma\cdots)/\cos\alpha\cos\beta\cos\gamma\cdots$$
$$=1-\Sigma\tan\alpha\tan\beta+\Sigma\tan\alpha\tan\beta\tan\gamma\tan\delta\cdots,$$

wherein $\Sigma\tan\alpha$ stands for the sum of the tangents of all the angles, $\Sigma\tan\alpha\tan\beta$ for the sum of their products taken two and two, and so on. Prove by induction.

COR. 2.
$$\sin 3\alpha=3\sin\alpha\cos^2\alpha-\sin^3\alpha=3\sin\alpha-4\sin^3\alpha,$$
$$\cos 3\alpha=\cos^3\alpha-3\cos\alpha\sin^2\alpha=-3\cos\alpha+4\cos^3\alpha,$$
$$\sin 4\alpha=4\sin\alpha\cos^3\alpha-4\sin^3\alpha\cos\alpha$$
$$=4\sin\alpha\cos\alpha-8\sin^3\alpha\cos\alpha,$$
$$\cos 4\alpha=\cos^4\alpha-6\sin^2\alpha\cos^2\alpha+\sin^4\alpha$$
$$=1-8\cos^2\alpha+8\cos^4\alpha,$$
$$\sin n\alpha=n\sin\alpha\cos^{n-1}\alpha$$
$$-\tfrac{1}{6}n(n-1)(n-2)\sin^3\alpha\cos^{n-3}\alpha+\cdots,$$
$$\cos n\alpha=\cos^n\alpha-\tfrac{1}{2}n(n-1)\sin^2\alpha\cos^{n-2}\alpha+\cdots,$$

wherein the coefficients in the value of sin $n\alpha$ are those of the second, fourth $\cdots$ terms of the expansion of $(a+b)^n$; and those in the value of cos $n\alpha$ are the first, third $\cdots$ terms of the same expansion. Prove by induction.

QUESTIONS.

1. If α, β, γ be any three plane angles, then:

$$-\sin(\alpha+\beta+\gamma)+\sin(-\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma)$$
$$+\sin(\alpha+\beta-\gamma)=4\sin\alpha\sin\beta\sin\gamma.$$
$$\cos(\alpha+\beta+\gamma)+\cos(-\alpha+\beta+\gamma)+\cos(\alpha-\beta+\gamma)$$
$$+\cos(\alpha+\beta-\gamma)=4\cos\alpha\cos\beta\cos\gamma.$$

If $\text{A}+\text{B}+\text{C}=2\text{R}$, then:

2. $\tan\frac{1}{2}\text{A}\tan\frac{1}{2}\text{B}+\tan\frac{1}{2}\text{B}\tan\frac{1}{2}\text{C}+\tan\frac{1}{2}\text{C}\tan\frac{1}{2}\text{A}=1.$

3. $\cot\text{A}+\cot\text{B}+\cot\text{C}=\cot\text{A}\cot\text{B}\cot\text{C}+\csc\text{A}\csc\text{B}\csc\text{C}.$

4. $\tan\text{A}+\tan\text{B}+\tan\text{C}=\tan\text{A}\tan\text{B}\tan\text{C}.$

5. $\tan 3\alpha=(\sin\alpha+\sin 3\alpha+\sin 5\alpha)/(\cos\alpha+\cos 3\alpha+\cos 5\alpha).$

6. If α, β be any two plane angles, and n any integer, then:

$$[\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\sin(\alpha+3\beta)+\cdots$$
$$+\sin(\alpha+\overline{n-1}\beta)]\cdot 2\sin\tfrac{1}{2}\beta$$
$$=\cos(\alpha-\tfrac{1}{2}\beta)-\cos(\alpha+\overline{n-\tfrac{1}{2}}\beta),$$
$$[\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cos(\alpha+3\beta)+\cdots$$
$$+\cos(\alpha+\overline{n-1}\beta)]\cdot 2\sin\tfrac{1}{2}\beta$$
$$=\sin(\alpha+\overline{n-\tfrac{1}{2}}\beta)-\sin(\alpha-\tfrac{1}{2}\beta).$$

7. From the results of ex. 6, prove that: [n any pos. integer.

$$\sin\alpha+\sin(\alpha+4\text{R}/n)+\cdots+\sin[\alpha+4\text{R}(n-1)/n]=0,$$
$$\cos\alpha+\cos(\alpha+4\text{R}/n)+\cdots+\cos[\alpha+4\text{R}(n-1)/n]=0.$$

8. In the results of ex. 7, take $n=3$, and prove that:

$$\sin\alpha+\sin\overline{60^\circ-\alpha}-\sin\overline{60^\circ+\alpha}=0,$$
$$\cos\alpha-\cos\overline{60^\circ-\alpha}-\cos\overline{60^\circ+\alpha}=0.$$

9. In the results of ex. 7, take $n=5$, and prove that:

$$\sin\alpha+\sin\overline{72^\circ+\alpha}+\sin\overline{36^\circ-\alpha}-\sin\overline{36^\circ+\alpha}-\sin\overline{72^\circ-\alpha}=0,$$
$$\cos\alpha+\cos\overline{72^\circ+\alpha}-\cos\overline{36^\circ-\alpha}-\cos\overline{36^\circ+\alpha}+\cos\overline{72^\circ-\alpha}=0.$$

10. Show that when $n=3$ the formula found in ex. 7 verifies the sines and cosines of all angles in the first quarter, if to α be given values from 0° to 30°.

11. In the results of ex. 7, take $n=9, 15, 25, 27, 45$, in turn, and thence find other formulæ of verification.

§11. INVERSE FUNCTIONS.

If a be a number, and α an angle such that $a=\sin\alpha$, this relation is expressed by the equation $\alpha=\sin^{-1}a$, which is read α is the *anti-sine* of a. So, the equation $\beta=\cos^{-1}b$ means that β is an angle whose cosine is b, and $\gamma=\tan^{-1}c$, that γ is an angle whose tangent is c.

It is to be noted that, while the equations $a=\sin\alpha$, $b=\cos\beta$, $c=\tan\gamma$, give a, b, c single values for single values of α, β, γ, the equations $\alpha=\sin^{-1}a$, $\beta=\cos^{-1}b$, $\gamma=\tan^{-1}c$ give α, β, γ many values for single values of a, b, c: for α, $2\text{R}-\alpha$, and all the congruents of these angles have the same sine; $\beta, -\beta$, and all their congruents have the same cosine; and γ, $2\text{R}+\gamma$, and all their congruents have the same tangent.

E.g. $\sin^{-1}\frac{1}{2}=30°,\ 150°,\ 390°,\ 510° \cdots\ -210°,\ -330° \cdots$.

So, $\cos^{-1}\sqrt{\frac{1}{2}}=45°,\ -45°,\ 315°,\ -315° \cdots$.

So, $\tan^{-1}\sqrt{3}=60°,\ 240°,\ 420°,\ 600° \cdots\ -120°,\ -300°, \cdots$.

Many of the theorems of trigonometry may be expressed in terms of inverse functions; and sometimes with advantage.

E.g. if x, y, z stand for the sines of the angles α, β, γ,

then $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$, may be written

$$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}[x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}].$$

So, $\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$
$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma$, may be written

$$\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\sin^{-1}[x\sqrt{(1-y^2-z^2+y^2z^2)}$$
$$+y\sqrt{(1-z^2-x^2+z^2x^2)}+z\sqrt{(1-x^2-y^2+x^2y^2)}-xyz].$$

So, $\sin 2\alpha=2\sin\alpha\cos\alpha$, may be written

$$2\sin^{-1}x=\sin^{-1}[2x\sqrt{(1-x^2)}].$$

So, $\sin\frac{1}{2}\alpha=\sqrt{\frac{1}{2}(1-\cos\alpha)}$ may be written

$$\tfrac{1}{2}\sin^{-1}x=\sin^{-1}\sqrt{\tfrac{1}{2}[1-\sqrt{(1-x^2)}]}.$$

These relations are always true:

$$\sin^{-1}x=\csc^{-1}1/x,\quad \cos^{-1}x=\sec^{-1}1/x,\quad \tan^{-1}x=\cot^{-1}1/x,$$
$$\sin^{-1}x+\cos^{-1}x=\text{R},\quad \sec^{-1}x+\csc^{-1}x=\text{R},\quad \tan^{-1}x+\cot^{-1}x=\text{R}.$$

QUESTIONS.

Translate these formulæ into inverse forms :

1. $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$.
2. $\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha \tan\beta)$.
3. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$.
4. $\tan 2\alpha = 2\tan\alpha/(1-\tan^2\alpha)$.
5. $\cos\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1+\cos\alpha)}$.
6. $$\begin{aligned}\tan\tfrac{1}{2}\alpha &= \sin\alpha/(1+\cos\alpha)\\ &= (1-\cos\alpha)/\sin\alpha\\ &= \sqrt{[(1-\cos\alpha)/(1+\cos\alpha)]}.\end{aligned}$$
7. $\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma$ $-\sin\beta\sin\gamma\cos\alpha - \sin\gamma\sin\alpha\cos\beta$.
8. $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}.$$
9. $\cos 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha$.
10. $\tan 3\alpha = (3\tan\alpha - \tan^3\alpha)/(1-3\tan^2\alpha)$.

Show that

11. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \text{R}$; $\cos^{-1}\frac{5}{13} + \cos^{-1}\frac{12}{13} = \text{R}$; $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{4}{3} = \text{R}$.
12. $\sin(3\sin^{-1}x) = 3x - 4x^3$. [$x$ any proper fraction.
    $\cos(3\cos^{-1}x) = -3x + 4x^3$.
    $\tan(3\tan^{-1}x) = (3x - x^3):(1-3x^2)$. [$x$ any number.
13. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{1}{2}\text{R}$. [Euler.
14. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{1}{2}\text{R}$. [Dase.
15. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{1}{2}\text{R}$. [Hutton.
16. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{1}{2}\text{R}$. [Machin.
17. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{1}{2}\text{R}$. [Rutherford.
18. $5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79} = \frac{1}{2}\text{R}$. [Euler.

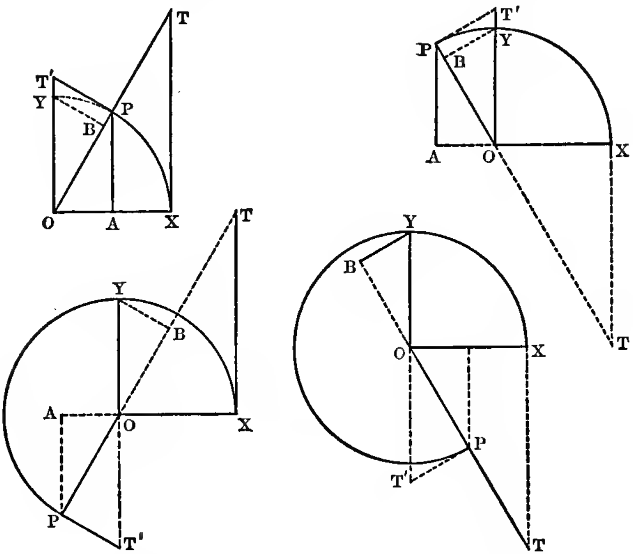
Solve the equations :

19. $\sin^{-1}3x + \sin^{-1}4x = \text{R}$.
20. $\tan^{-1}2x + \tan^{-1}3x = \frac{1}{2}\text{R}$.

§ 12. GRAPHIC REPRESENTATION OF TRIGONOMETRIC RATIOS.

Let XP be any arc with centre O and radius OX, and let PY be the arc complementary to XP;

through P, X draw AP, XT normal to OX, and through Y, P draw BY, PT′ normal to OP, with T on OP and T′ on OY;



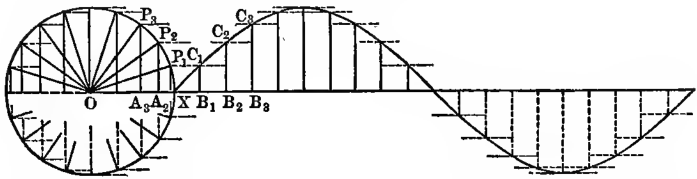
then AP, OA, XT, OT are the *sine, cosine, tangent,* and *secant* of the arc XP;

and BY, OB, PT′, OT′ are the *sine, cosine, tangent,* and *secant* of the complementary arc PY, and the *cosine, sine, cotangent,* and *cosecant* of the arc XP.

These lines are called *line-functions* of arcs as distinguished from the *ratio-functions* of angles; and if they be divided by the radius, the ratios so found are the ratio-functions heretofore defined. With arcs of the same radius the ratios of their line-functions are equal to the ratios of the like ratio-functions of their angles.

CURVE OF SINES.

Let OX be the radius of a circle, and divide the circumference into any convenient parts at P_1, $P_2 \cdots$;



draw A_1P_1, $A_2P_2 \cdots$ normal to OX, and sines of the arcs XP_1, $XP_2 \cdots$;

upon OX lay off XB_1, $XB_2 \cdots$ equal to the arcs XP_1, $XP_2 \cdots$;

at B_1, $B_2 \cdots$ erect perpendiculars to OX and take C_1, $C_2 \cdots$ such that $B_1C_1 = A_1P_1$, $B_2C_2 = A_2P_2 \cdots$;

through C_1, $C_2 \cdots$ draw a smooth curve; it is the *curve of sines*, and the following relations are manifest:

The sine is 0 for the angle 0;

is nearly as long as the arc for a small angle;

increases more and more slowly;

is equal to the radius, and its ratio is $+1$, its maximum, for a right angle;

decreases, at first slowly, but faster and faster as the angle approaches two right angles;

is 0 for two right angles;

decreases from 0 to the opposite of the radius, and its ratio is -1, its minimum, as the angle grows from two right angles to three;

increases to 0 as the angle grows from three right angles to four;

is again 0 at the end of the first revolution; and so on.

The sine has all values between the radius and its opposite.

If the revolution be continuous, the values of the sine are periodic, every successive revolution indicating a new cycle and a new wave in the curve. The sines are equal for pairs of angles symmetric about the normal at O.

OTHER TRIGONOMETRIC CURVES.

The tangent is 0 for the angle 0;
increases through the first quarter to $+\infty$; leaps to $-\infty$;
increases through the second quarter to 0;
increases through the third quarter to $+\infty$; leaps to $-\infty$;
increases through the fourth quarter to 0; and so on.

The tangent has all values from $-\infty$ to $+\infty$. Tangents are equal for pairs of angles that differ by a half revolution.

The secant is equal to the radius, and its ratio is $+1$ for the angle 0;
increases through the first quarter to $+\infty$; leaps to $-\infty$;
increases through the second quarter to the opposite of the radius, and its ratio is -1;
decreases through the third quarter to $-\infty$; leaps to $+\infty$;
decreases through the fourth quarter to the value at the beginning; and so on.

The secant has no value smaller than the radius. Secants are equal for pairs of angles symmetric as to the initial line.

The cosine, cotangent, cosecant have the same bounds as the sine, tangent, secant; they go through like changes and are represented by like curves; but they begin, for the angle 0, with different values, viz., the radius, ∞, ∞.

QUESTIONS.

1. Show directly from the definitions what are the largest and what the smallest values that each function may have, and state for what angles the several functions take these values.

So, what are the greatest and what the least values.

2. Draw the curve of tangents, curve of secants, curve of cosines, curve of cotangents, and curve of cosecants.

3. Trace the changes, when α increases from 0 to 4R, in:

$$\sin\alpha+\cos\alpha,\quad \tan\alpha+\cot\alpha,\quad \sin\alpha+\csc\alpha,$$
$$\sin\alpha-\cos\alpha,\quad \tan\alpha-\cot\alpha,\quad \sin\alpha-\csc\alpha.$$

QUESTIONS FOR REVIEW.

1. Find $\sec(\alpha\pm\beta)$, $\csc(\alpha\pm\beta)$ in the ratios of α and β.

2. Given $\tan 1^\circ 30' = .0262$: find $\tan 21^\circ$, $\tan 24^\circ$, $\cot 21^\circ$, $\cot 24^\circ$.

Show that:

3. $\cos 2\alpha = 2(\sin\alpha + \frac{1}{2}\text{R})(\sin\alpha + \frac{3}{2}\text{R})$.

4. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$
   $-\sin(\alpha+\beta+\gamma) = 4\sin\alpha\sin\beta\sin\gamma$.

5. $\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)+\cos^2(\alpha-\beta)$
   $=1+2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$.

6. $\tan\alpha+\tan\beta = \sin(\alpha+\beta)/\cos\alpha\cos\beta$.

7. $\tan\frac{1}{2}(\alpha+\beta) = (\sin\alpha+\sin\beta)/(\cos\alpha+\cos\beta)$.

Solve these equations:

8. $4\sin\theta\sin 3\theta = 1$.

9. $\sin 3\theta - \sin\theta = 0$.

10. $\tan\theta + \tan 2\theta = \tan 3\theta$.

11. $\cos\theta - \sin\theta = \sqrt{\frac{1}{2}}$.

12. $3\cos\theta + \sin\theta = 2$.

Trace the changes in sign and magnitnde as θ grows from 0 to 4R, in:

13. $\cos 2\theta/\cos\theta$.

14. $\sin\theta - \sin\frac{1}{2}\theta$.

15. $\tan\theta + \cot\theta$.

16. $\sin\theta + \sin 2\theta + \sin 4\theta$.

Prove the eqnations:

17. $\tan^{-1}\frac{2}{11} + 2\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{2}$.

18. $\cot^{-1}2 + \csc^{-1}\sqrt{10} = \frac{1}{2}\text{R}$.

19. $\sin^{-1}x + \tan^{-1}(1-x) = 2\tan^{-1}\sqrt{(x-x^2)}$.

20. $\sin^{-1}[2x/(1+x^2)] + \tan^{-1}[2x/(1-x^2)] = \text{R}$.

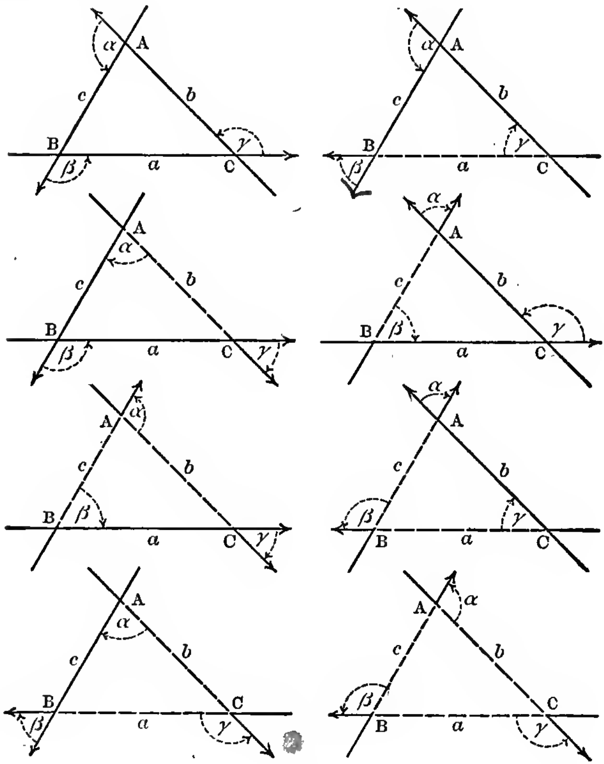
21. If $\tan\frac{1}{2}\theta = \tan^3\frac{1}{2}\phi$, and $\tan\phi = 2\tan\alpha$, then $\overline{\theta+\phi} = 2\alpha$.

22. If $\sin(x+\alpha)/\sin(x+\beta) = \sqrt{(\sin 2\alpha/\sin 2\beta)}$,
then $\tan^2 x = \tan\alpha\tan\beta$.

III. PLANE TRIANGLES.

§ 1. THE GENERAL TRIANGLE.

Let *a, b, c* be any three directed lines that meet each other, *b, c* at A, *c, a* at B, *a, b* at C, the figure so formed is a plane triangle, ABC.



Of the eight figures shown here, the first may be called the *ideal triangle:* its sides, taken in order, and followed in their positive directions each till it crosses the next one, form a

closed figure, and the primary angles *bc, ca, ab* are all positive. In going round the other figures from vertex to vertex in order, some of the sides must be followed in their negative directions and some of the angles are negative. Such triangles may be called *deformed triangles.*

The eight figures show all the possible ways of describing a plane triangle: for each side must be traversed in one of two ways, forward or backward; and the two ways of describing the first side may be combined at will with the two ways of describing the second side, and these $2 \cdot 2$ ways, with the two ways of describing the third side, making $2 \cdot 2 \cdot 2$ ways of describing the three sides.

If α, β, γ stand for the exterior angles of the triangle, *i.e.* for the angles *bc, ca, ab*, then, in the ideal triangle, α, β, γ are the supplements of the interior angles, A, B, C, commonly called the angles of the triangle, and their sum is four right angles. In the deformed triangles the sum of α, β, γ is some congruent of four right angles.

QUESTIONS.

1. What is the effect on the values of the sides and angles of an ideal triangle, of reversing the direction of one of the bounding lines? of reversing two of the bounding lines? of reversing all three of them? of keeping the directions fixed and moving one bounding line parallel to itself, to a position equally distant from, and on the other side of, the opposite vertex? of turning over the plane of the triangle?

2. If a man, walking around a triangular field, ABC, start at A, walk to B, turn to the left so as to face C, walk to C, turn to the left so as to face A, walk to A, turn to the left so as to face B, through what angle has he turned?

3. So, if, starting from A and going about the field, he face in the direction BA, and walk backward from A to B having the field on his right, then, facing in the direction CB, walk backward to C, then, facing in the direction AC, walk backward to A, through what angle has he turned?

§2. GENERAL PROPERTIES OF PLANE TRIANGLES.

The discussions below apply directly to the ideal triangle, but with due attention to signs they apply to the deformed triangles as well.

The letters a, b, c have a double use: first as the names of the indefinite directed bounding lines of the triangle, second as the segments BC, CA, AB, of these bounding lines. These segments are the *sides* of the triangle.

E.g. in the statement α is the angle bc the indefinite lines b, c are meant and α is their angle;

but in the equation $a^2 = b^2 + c^2 + 2\,bc \cos \alpha$ a, b, c are the measured and directed sides.

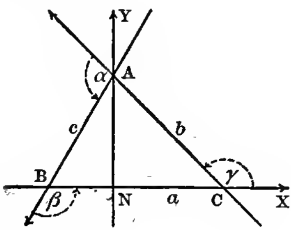
The context always shows clearly which use is intended.

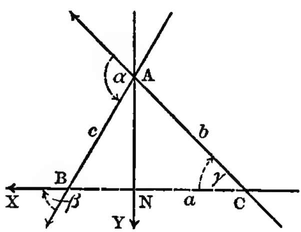
LAW OF COSINES.

THEOR. 1. *If a, b, c be the sides of a plane triangle, and α, β, γ the angles bc, ca, ab, then:*

$$a^2 = b^2 + c^2 + 2\,bc \cos \alpha,$$
$$b^2 = c^2 + a^2 + 2\,ca \cos \beta,$$
$$c^2 = a^2 + b^2 + 2\,ab \cos \gamma.$$

For, project the closed broken line $a+b+c$ on a;





then $\because$ $\angle ab = \gamma$, $\angle ac = -\beta$,

$\therefore\ a + b \cos \gamma + c \cos(-\beta) = 0,$ [II, theor. 10.

i.e. $a + b \cos \gamma + c \cos \beta = 0.$ [II, theor. 5.

So, $b + c \cos \alpha + a \cos \gamma = 0,$ [project $a+b+c$ on b.

and $c + a \cos \beta + b \cos \alpha = 0.$ [project $a+b+c$ on c.

Multiply the first of these equations by $-a$, the second by b, the third by c, and add;

then $\quad -a^2+b^2+c^2+2bc\cos\alpha=0, \quad$ *i.e.* $\quad a^2=b^2+c^2+2bc\cos\alpha$;

For the second formula multiply by $a, -b, c$ and add, and for the third multiply by $a, b, -c$ and add.

COR. 1. *cos* $\frac{1}{2}\alpha=\sqrt{[(s-b)(s-c)/bc]}$. $\qquad [2s=a+b+c$.

For $\because 2\cos^2\frac{1}{2}\alpha=1+\cos\alpha$

$$\begin{aligned}
&=1+(a^2-b^2-c^2)/2bc\\
&=(a^2-b^2+2bc-c^2)/2bc\\
&=[a^2-(b-c)^2]/2bc\\
&=(a-b+c)(a+b-c)/2bc\\
&=4(s-b)(s-c)/2bc,
\end{aligned}$$

$\therefore \cos\frac{1}{2}\alpha=\sqrt{[(s-b)(s-c)/bc]}$. Q.E.D.

COR. 2. *sin* $\frac{1}{2}\alpha=\sqrt{[s(s-a)/bc]}$.

For $\because 2\sin^2\frac{1}{2}\alpha=1-\cos\alpha$

$$\begin{aligned}
&=1-(a^2-b^2-c^2)/2bc\\
&=(b^2+2bc+c^2-a^2)/2bc\\
&=[(b+c)^2-a^2]/2bc\\
&=(b+c+a)(b+c-a)/2bc\\
&=4s(s-a)/2bc,
\end{aligned}$$

$\therefore \sin\frac{1}{2}\alpha=\sqrt{[s(s-a)/bc]}$. Q.E.D.

COR. 3. *cot* $\frac{1}{2}\alpha=\sqrt{[(s-b)(s-c)/s(s-a)]}$.

COR. 4. *If $a, b, c, \alpha, \beta, \gamma$ be the parts of an ideal triangle, and if* A, B, C *be the interior angles of the triangle, then:*

$$\begin{aligned}
&\cos A=(b^2+c^2-a^2)/2bc,\\
&\sin\tfrac{1}{2}A=\sqrt{[(s-b)(s-c)/bc]},\\
&\cos\tfrac{1}{2}A=\sqrt{[s(s-a)/bc]},\\
&\tan\tfrac{1}{2}A=\sqrt{[(s-b)(s-c)/s(s-a)]}.
\end{aligned}$$

For $\because \alpha$, A are supplementary angles,

$\therefore \frac{1}{2}\alpha, \frac{1}{2}A$ are complementary angles,

and $\quad \cos\alpha=-\cos A, \quad \cos\frac{1}{2}\alpha=\sin\frac{1}{2}A,$

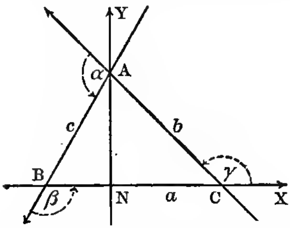
$\quad \sin\frac{1}{2}\alpha=\cos\frac{1}{2}A, \quad \cot\frac{1}{2}\alpha=\tan\frac{1}{2}A.$

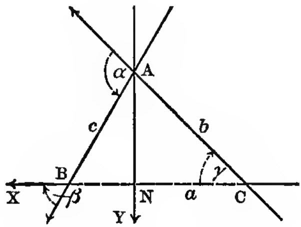
LAW OF SINES.

THEOR. 2. *If a, b, c be the sides of a plane triangle, and α, β, γ be the angles bc, ca, ab, then:*

$$a/\sin\alpha = b/\sin\beta = c/\sin\gamma.$$

For, draw any normal to a and project the closed broken line $a+b+c$ on this normal;





then $\because \angle ab = \gamma$, $\angle ac = -\beta$,

and the projection of a on its normal is naught,

$\therefore 0 + b\sin\gamma + c\sin(-\beta) = 0$,

$\therefore b\cdot\sin\gamma = c\cdot\sin\beta$,

$\therefore b/\sin\beta = c/\sin\gamma$.

So, $c/\sin\gamma = a/\sin\alpha$, [project $a+b+c$ on a normal to b.

and $a/\sin\alpha = b/\sin\beta$. [project $a+b+c$ on a normal to c.

COR. 1. $(a+b)/c = \cos\frac{1}{2}(\alpha-\beta)/\cos\frac{1}{2}\gamma$.

$(a-b)/c = -\sin\frac{1}{2}(\alpha-\beta)/\sin\frac{1}{2}\gamma$.

For $\because a/c = \sin\alpha/\sin\gamma$, $b/c = \sin\beta/\sin\gamma$, [above.

$\therefore (a+b)/c = (\sin\alpha + \sin\beta)/\sin\gamma$

$= 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)/2\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma$

$= \cos\frac{1}{2}(\alpha-\beta)/\cos\frac{1}{2}\gamma$. [$\frac{1}{2}(\alpha+\beta) = \sup\frac{1}{2}\gamma$.

So, $(a-b)/c = (\sin\alpha - \sin\beta)/\sin\gamma$

$= 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)/2\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma$

$= -\sin\frac{1}{2}(\alpha-\beta)/\sin\frac{1}{2}\gamma$.

COR. 2. $(a-b)/(a+b) = -\tan\frac{1}{2}(\alpha-\beta)/\tan\frac{1}{2}\gamma$.

COR. 3. *If $a, b, c, \alpha, \beta, \gamma$ be the parts of an ideal triangle, and if* A, B, C *be the interior angles of the triangle, then :*

$$a/\sin \mathrm{A} = b/\sin \mathrm{B} = c/\sin \mathrm{C},$$
$$(a+b)/c = \cos \tfrac{1}{2}(\mathrm{A}-\mathrm{B})/\sin \tfrac{1}{2}\mathrm{C},$$
$$(a-b)/c = \sin \tfrac{1}{2}(\mathrm{A}-\mathrm{B})/\cos \tfrac{1}{2}\mathrm{C},$$
$$(a-b)/(a+b) = \tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B})/\cot \tfrac{1}{2}\mathrm{C}.$$

For $\because$ α, A are supplementary angles, and so are β, B and γ, C,

$$\therefore \tfrac{1}{2}(\alpha-\beta) = -\tfrac{1}{2}(\mathrm{A}-\mathrm{B}),$$

and $\tfrac{1}{2}\gamma$, $\tfrac{1}{2}\mathrm{C}$ are complementary,

$$\therefore \sin\alpha = \sin \mathrm{A}, \quad \sin\beta = \sin \mathrm{B}, \quad \sin\gamma = \sin \mathrm{C},$$
$$\cos\tfrac{1}{2}\gamma = \sin\tfrac{1}{2}\mathrm{C}, \quad \sin\tfrac{1}{2}\gamma = \cos\tfrac{1}{2}\mathrm{C}, \quad \cot\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}\mathrm{C},$$
$$\tan\tfrac{1}{2}(\alpha-\beta) = -\tan\tfrac{1}{2}(\mathrm{A}-\mathrm{B}).$$

QUESTIONS.

1. What is the value, in terms of a, b, c, of :

$\cos\beta$, $\cos\tfrac{1}{2}\beta$, $\sin\tfrac{1}{2}\beta$, $\cot\tfrac{1}{2}\beta$?

$\cos \mathrm{B}$, $\sin\tfrac{1}{2}\mathrm{B}$, $\cos\tfrac{1}{2}\mathrm{B}$, $\tan\tfrac{1}{2}\mathrm{B}$?

$\cos\gamma$, $\cos\tfrac{1}{2}\gamma$, $\sin\tfrac{1}{2}\gamma$, $\cot\tfrac{1}{2}\gamma$?

$\cos \mathrm{C}$, $\sin\tfrac{1}{2}\mathrm{C}$, $\cos\tfrac{1}{2}\mathrm{C}$, $\tan\tfrac{1}{2}\mathrm{C}$?

2. What signs are to be given to the radicals in theor. 1, cors. 1, 2, 3, in case of an ideal triangle ?

3. Show that the values of $\cos\tfrac{1}{2}\alpha$, $\sin\tfrac{1}{2}\alpha$, $\cdots$ are impossible if one side be greater than the sum of the other two.

In any plane triangle ABC :

4. $\cos\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}\mathrm{B}/\sin\tfrac{1}{2}\mathrm{C} = s/c.$

5. $\cos\tfrac{1}{2}\mathrm{A}\sin\tfrac{1}{2}\mathrm{B}/\cos\tfrac{1}{2}\mathrm{C} = (s-a)/c.$

6. $\sin\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}\mathrm{B}/\cos\tfrac{1}{2}\mathrm{C} = (s-b)/c.$

7. $\sin\tfrac{1}{2}\mathrm{A}\sin\tfrac{1}{2}\mathrm{B}/\sin\tfrac{1}{2}\mathrm{C} = (s-c)/c.$

8. $a\cos \mathrm{B} + b\cos \mathrm{A} = c$; $\quad a\cos \mathrm{B} - b\cos \mathrm{A} = (a^2-b^2)/c.$

9. $a\cos \mathrm{B}\cos \mathrm{C} + b\cos \mathrm{C}\cos \mathrm{A} + c\cos \mathrm{A}\cos \mathrm{B}$
$$= a\sin \mathrm{B}\sin \mathrm{C} = b\sin \mathrm{C}\sin \mathrm{A} = c\sin \mathrm{A}\sin \mathrm{B}.$$

10. $a\cos \mathrm{A} + b\cos \mathrm{B} + c\cos \mathrm{C} = 2a\sin \mathrm{B}\sin \mathrm{C} = \cdots.$

11. $a\sin(\mathrm{B}-\mathrm{C}) + b\sin(\mathrm{C}-\mathrm{A}) + c\sin(\mathrm{A}-\mathrm{B}) = 0$

§ 3. SOLUTION OF PLANE TRIANGLES.

PROB. 1. TO SOLVE AN OBLIQUE PLANE TRIANGLE.

Apply such of the formulæ of theors. 1, 2, *and their corollaries, as serve to express the values of the unknown parts in terms of the known parts.*

CHECK : *Form an equation involving the three computed parts ; but use no part in the same way in the solution and the check.*

Cases of the general triangle appear in discussing the relations of coplanar forces in mechanics, in particular when one of the forces is the resultant of the other two ; and in the solution of such triangles, the general formulæ given above may be used. For ordinary purposes the ideal triangle alone is sufficient, and in its solution it is convenient and in accord with usage to ignore the exterior angles α, β, γ, and to use the interior angles A, B, C. The rules may then take the form shown below. There are four cases.

(*a*) *Given a, b, c, the three sides :*

then $\cos A = (b^2 + c^2 - a^2)/2\,bc$,

$\cos B = (c^2 + a^2 - b^2)/2\,ca$,

$\cos C = (a^2 + b^2 - c^2)/2\,ab$; *check :* $A + B + C = 2\,R$.

These formulæ are used if a, b, c be expressed in numbers so small that the squares, sums, and quotients are easily computed ; and the angles are then found from their natural cosines. If a, b, c be expressed in large numbers use the formulæ shown below, which are specially adapted to logarithmic work.

$$\tan\tfrac{1}{2}A = \sqrt{[(s-a)\,(s-b)\,(s-c)/s]}/(s-a),$$

$$\tan\tfrac{1}{2}B = \sqrt{[(s-a)\,(s-b)\,(s-c)/s]}/(s-b),$$

$$\tan\tfrac{1}{2}C = \sqrt{[(s-a)\,(s-b)\,(s-c)/s]}/(s-c).$$

For
$$\begin{aligned}\tan\tfrac{1}{2}A &= \sqrt{[(s-b)\,(s-c)/s\,(s-a)]}\\ &= \sqrt{[(s-b)\,(s-c)\,(s-a)/s(s-a)^2]}\\ &= \sqrt{[(s-a)(s-b)\,(s-c)/s]}/(s-a),\end{aligned}$$

and so for $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$.

The special advantage of these formulæ lies in this, that the radical part is the same for each of the three half angles.

E.g. Let a, b, c be 3, 5, 7; then, using the upper formulæ, the work may take this form:

$$\cos A = (25+49-9)/70 = 65/70 = .9286, \text{ and } A = 21^\circ 47'$$
$$\cos B = (49+9-25)/42 = 33/42 = .7857, \text{ and } B = 38^\circ 13'$$
$$\cos C = (9+25-49)/30 = -15/30 = -.5000, \text{ and } C = 120^\circ$$

check: $A+B+C=180^\circ$. $\quad 180^\circ$

So, let a, b, c be 357, 573, 735; then, using the lower formulæ, the work may take this form:

357	$s = 832.5$	log,	2.9204 −
573	$s-a = 475.5$		2.6772 +
735	$s-b = 259.5$		2.4141 +
2)1665	$s-c = 97.5$		1.9890 +
832.5	*check*: 1665.		2)4.1599
			2.0800

2.0800	2.0800	2.0800
−2.6772	−2.4141	−1.9890
log-tan $\frac{1}{2}A = 9.4028$	log-tan $\frac{1}{2}B = 9.6659$	log-tan $\frac{1}{2}C = 0.0910$
$\frac{1}{2}A = 14^\circ 11'$	$\frac{1}{2}B = 24^\circ 51\frac{1}{2}'$	$\frac{1}{2}C = 50^\circ 58'$
$A = 28^\circ 22'$	$B = 49^\circ 43'$	$C = 101^\circ 56'$

check: $A+B+C=180^\circ$ nearly.

QUESTIONS.

1. Show by the formulæ that a triangle is possible only when each side is less than the sum of the other two sides.

What sign must be given to the radical in an ideal triangle?

2. Solve the triangle, given a, 127 m.; b, 64.9 m.; c, 152.16 m.
[55° 19.4′, 24° 51.1′, 99° 49.2′.

3. Solve the triangle; given a, 659.7; b, 318.2; c, 527.6.

4. Solve the triangle, given a, 625; b, 615; c, 11.

Before solving show which of the angles A, B, C are large, which small, and which smallest.

Can an exact solution be made?

(*b*) *Given* A, B, *c*, *two angles and a side:*

then $C = 180° - (A + B)$, $a = \sin A \cdot c/\sin C$, $b = \sin B \cdot c/\sin C$.

check: $\sin \frac{1}{2}C = \sqrt{[(s-a)(s-b)/ab]}$.

E.g. let A, B, *c* be 50°, 75°, 120 yards; then the work may take this form:

$C = 180° - 125° = 55°$

$\log 120 = 2.0792$	2.1658	2.1658
$\text{log-sin } 55° = 9.9134$	$\text{log-sin } 50° = 9.8843$	$\text{log-sin } 75° = 9.9849$
2.1658	$\log a = 2.0501$	$\log b = 2.1507$
	$a = 112.2$ yards.	$b = 141.5$ yards.

check:

$c = 120$		
$a = 112.2$	log, $2.0501-$	
$b = 141.5$	$2.1507-$	
$2)\overline{373.7}$		
$s = 186.85$		
$s-a = 74.65$	$1.8730+$	
$s-b = 45.35$	$1.6566+$	
$s-c = 66.85$	$2)\overline{9.3238}$	$\frac{1}{2}C = 27° 30'$
373.7	9.6644	$\text{log-sin } \frac{1}{2}C = 9.6644$

QUESTIONS.

1. In examples under this case, is there always a solution? Is there ever more than one solution? What limitations are there on the values of the two given angles A, B?

2. Solve the triangle, given A, 34°; B, 95°; *c*, 1389 ft.
[51°, 9.995, 17.805.

3. Solve the triangle, given B, 58° 30′; C, 120° 13′; *a*, 5387 yds.
Can an exact solution be made with the angles B, C so large, and A so small?

4. Write out the formulæ for the solution and the check when B, C, *a* are given.
So, when C, A, *b* are given.
So, when A, B, *a* are given.

5. Why may not more than three parts be given?
E.g. Why may not the data be A, 50°; B, 75°; *a*, 20; *b*, 30?

(*c*) *Given* a, b, C, *two sides and their angle:*

then $\frac{1}{2}(A+B)=90°-\frac{1}{2}C$, $\tan\frac{1}{2}(A-B)=\cot\frac{1}{2}C\cdot(a-b)/(a+b)$,

$\frac{1}{2}(A+B)+\frac{1}{2}(A-B)=A$, $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B$,

$c=\sin C\cdot a/\sin A$.

check: $(b+c)/a=\cos\frac{1}{2}(B-C)/\sin\frac{1}{2}A$.

E.g. let a, b, C be 635, 361, 61° 17′; then the work may take this form:

FORMULÆ.	NUMBERS.	LOGARITHMS.
$\cot\frac{1}{2}C$	30° 38′	0.2275 +
$\cdot a-b$	274	2.4378 +
$:a+b$	996	2.9983 −
$=\tan\frac{1}{2}(A-B)$	24° 54½′	9.6670
$90°-\frac{1}{2}C$		
$=\frac{1}{2}(A+B)$	59° 21½′	
A (sought)	84° 16′	
B (sought)	34° 27′	
a	635	2.8028 +
$\cdot\sin C$	61° 17′	9.9430 +
$:\sin A$	84° 16′	9.9978 −
$=c$ (sought)	559.75	2.7480
check: $b+c$	920.75	2.9642 +
$:a$	635	2.8028 −
		0.1614
$=\cos\frac{1}{2}(C-B)$	13° 25′	9.9880 +
$:\sin\frac{1}{2}A$	42° 8′	9.8266 −
		0.1614

QUESTIONS.

1. In examples under this case, is there always a solution? Is there ever more than one solution? Are any limitations to be put upon the lengths of the sides or the magnitude of their angle? Between what limits do $\frac{1}{2}(A+B)$, $\frac{1}{2}(A-B)$ lie?

2. Solve the triangle, given a, 25.3; b, 136; C, 98° 15′.
[10° 10′, 71° 35′, 141.86.

(*d*) *Given* b, c, B, *two sides and an angle opposite one of them:* then $\sin C = c \cdot \sin B/b$, $A = 180° - (B + C)$, $a = \sin A \cdot b/\sin B$. *check:* $(a+b)/c = \cos\frac{1}{2}(A - B)/\sin\frac{1}{2}C$.

The angle C, found from the equation $\sin C = b \cdot \sin B/b$, may in general have two supplementary values [$\sin \sup \alpha = \sin \alpha$. and two triangles are then possible.

But there are some limitations:

1. If B, b, c be so related that $c \cdot \sin B > b$, then $\sin C > 1$, which is impossible, and there is no triangle.

2. If $c \cdot \sin B = b$, then $\sin C = 1$, C is a right angle, and there is one, a right triangle.

3. If either value of the angle C makes $B > C$ when $b \not> c$, or $B < C$ when $b \not< c$, that value must be rejected.

In particular: if B be acute, no triangle is possible if $b < p$, the perpendicular from A to the side a; one right triangle if $b = p$; two triangles if $p < b < c$; one, an isosceles triangle, if $b = c$; one triangle if $b > c$.

So, if B be right or obtuse, a triangle is possible only when $b > c$, and then but one.

QUESTIONS.

1. Draw figures to show the several cases outlined above, and show how the geometric constructions interpret the facts as shown by the formulæ, for the several cases.

Solve these triangles, given:

2. b, 18; c, 20; B, 55° 24′.
[66° 9′, 58° 27′, 18.64, or 113° 51′, 10° 45′, 4.08.

3. a, 10; b, 20; A, 30°. 4. b, 16; c, 20; B, 86° 40′.

5. c, 20; a, 20; C, 47° 9′. 6. a, 24; b, 20; A, 37° 36′.

7. a, 24; b, 20; A, 120°. [46° 12′, 13° 48′, 6.61.

8. a, 20; b, 20; A, 135°. 9. a, 16; b, 20; A, 150°.

10. Let O, P be two points 10 feet apart; about O describe a circle with radius 4 feet; through P draw a line making an angle of 20° with the line PO: at what distance from P does this line cut the circle?

QUESTIONS FOR REVIEW.

Solve these triangles, given :

1. a, 40 ; b, 50 ; c, 60.
2. a, 4 ; b, 5 ; c, 6.
3. a, 411 ; b, 522 ; c, 633.
4. A, 60° ; B, 60° ; c, 10.
5. a, 24 ; B, 45° ; C, 24°.
6. A, 31° 26′; b, 17.1 ; C, 47° 18′.
7. a, 14 ; b, 14 ; C, 60°.
8. a, 38.9 ; B, 9° 18′; c, 119.11.
9. A, 117° 23′; b, 6 ; c, 11.14.
10. a, 36 ; b, 40 ; A, 51° 16′.

11. If the three sides a, b, c of a triangle be given, find the length of the perpendiculars from the vertices upon the opposite sides ; of the lines connecting the vertices with the mid-points of the opposite sides ; of the segments of the bisectors of the angles, cut off by the opposite sides.

12. In ex. 10 of page 72, let the distance OP be a, the radius of the circle b, and the angle POQ, C : how many solutions are possible when $a > b$? when A = B ? when $a < b$?
Show how the angle C is limited in each of these cases.

13. Discuss ex. 12 if C be negative. So, a or b be negative.

14. The sides of a triangle are 3, 4, $\sqrt{38}$: show, without solving, that the largest angle is greater than 120°.

15. If a, b, c be in arithmetic progression, $3 \tan \frac{1}{2}\text{A} \cdot \tan \frac{1}{2}\text{C} = 1$.

16. If $\text{C} = 2\text{B}$, then $c^2 = b\,(a + b)$.

17. Show by trigonometry that if an angle of a triangle be bisected, the segments of the opposite side are proportional to the other two sides.

18. If $a \cos \text{A} = b \cos \text{B}$, the triangle is either right-angled or isosceles.

19. If P be any point in an equilateral triangle ABC, then

$$\cos(\text{BPC} - 60°) = (\text{PB}^2 + \text{PC}^2 - \text{PA}^2)/2\text{PB} \cdot \text{PC}.$$

20. Show how to solve a triangle from the three altitudes.

§ 4. SINES AND TANGENTS OF SMALL ANGLES.

If an angle be very small, its sine and tangent are also very small; but their logarithms are negative and very large, and they change rapidly and at rapidly varying rates. Such logarithms, therefore, are not convenient for use where interpolation is necessary, and in their stead the logarithms given below may be used; they are based on the following considerations:

An angle whose bounding arc is just as long as a radius is a *radian;* it is equal to $57° 17' 44.8''$, *i.e.* to $206264.8''$, and the number of seconds in an angle is 206264.8 times the number of radians. The index for radians is ^{r}.

For a small angle the number of radians in the bounding arc is a very small fraction, and it is a very little larger than the sine of the angle and a very little smaller than its tangent: it follows that, if a small angle be expressed in radians, the ratio $\sin A^r/A$ is a very little smaller, and the ratio $\tan A^r/A$ is a very little larger, than unity. These ratios approach unity closer and closer as the angle grows smaller.

If the angle be expressed in seconds, the ratio $\sin A''/A$ is a very little smaller than the reciprocal of 206264.8, and the ratio $\tan A''/A$ is a very little larger than this reciprocal. These ratios change very slowly, and hence interpolation is always possible; the table below gives their logarithms as far as 5°.

Angle.	log (sin A″/A).	Angle.	log (tan A″/A).	Angle.	log (tan A″/A).
0° –1° 4′	4.6856	0° –1° 18′	4.6856	3° 37′–3° 54′	4.6862
1° 5′ –2° 23′	4.6855	1° 19′–1° 59′	4.6857	3° 55′–4° 11′	4.6863
2° 24′–3° 11′	4.6854	2° –2° 29′	4.6858	4° 12′–4° 27′	4.6864
3° 12′–3° 50′	4.6853	2° 30′–2° 54′	4.6859	4° 28′–4° 41′	4.6865
3° 51′–4° 23′	4.6852	2° 55′–3° 16′	4.6860	4° 42′–4° 55′	4.6866
4° 24′–4° 52′	4.6851	3° 17′–3° 36′	4.6861	4° 55′–5° 00′	4.6867

The cosine and cotangent of an angle near 90° are the sine and tangent of the complementary small angle. The logarithm of the cotangent of a small angle is found by subtracting the modified logarithm of the tangent of the angle from 10; that of the tangent of an angle near 90°, by subtracting the modified logarithm of the tangent of the complementary small angle from 10.

TO TAKE OUT THE SINE OR TANGENT OF A SMALL ANGLE.

Take out the logarithm that corresponds to the number of degrees and minutes; and add the logarithm of the whole number of seconds in the angle.

Let A be the number of seconds in an angle;

then $\because \sin A'' = (\sin A''/A) \cdot A$,

$\therefore$ log-sin $A'' = \log(\sin A''/A) + \log A$;

and $\because \tan A'' = (\tan A''/A) \cdot A$,

$\therefore$ log-tan $A'' = \log(\tan A''/A) + \log A$.

E.g. log-sin $10'\ 30'' = \log(\sin 630''/630) + \log 630$
$= 4.6856 + 2.7993 = 7.4849$.

So, log-tan $3°\ 13'\ 40'' = \log(\tan 11620''/11620) + \log 11620$
$= 4.6860 + 4.0652 = 8.7512$.

The angle is found by a reverse process.

E.g. to take out log-sin^{-1} 8.4143:

From the table of sines and tangents, page xi, it appears that the angle sought lies just below $1°\ 30'$, and by the formula

$\log A =$ log-sin $A'' - \log(\sin A''/A)$;

and $\because 8.4143 - 4.6855 = 3.7288$,

$\therefore$ the angle is $5355''$; *i.e.* $1°\ 29'\ 15''$.

So, to take out log-sin^{-1} 8.8062:

The angle sought lies near $3°\ 40'$,

and $\because 8.8062 - 4.6853 = 4.1210$,

$\therefore$ the angle is $13212''$; *i.e.* $3°\ 40'\ 12''$.

QUESTIONS.

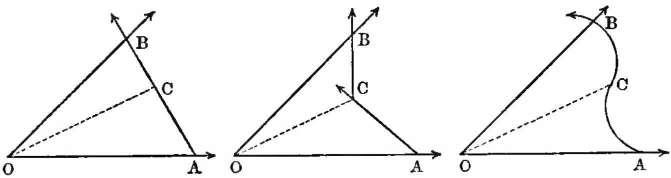
1. Find log-sin $22'$, $43'$, $1°\ 11'$, $1°\ 27'$, $2°\ 24'\ 36''$.
2. Find log-tan $22'$, $43'$, $1°\ 11'$, $1°\ 27'$, $2°\ 24'\ 36''$.
3. Find log-sin^{-1} 7.3146, 8.2719, 8.4185, 8.8927.
4. Find log-tan^{-1} 7.3146, 8.2719, 8.4185, 8.8927.

Solve these triangles, given:

5. a, 327; b, 328; c, 654. 6. a, 3279; b, 3280; c, 1°.

§ 5. DIRECTED AREAS.

If an elastic cord be stretched from a point O to a point A, and if while one end of this cord is fixed at O, the other end trace a line AB, straight, broken, or curved, the cord, now a radius vector of varying length, sweeps over the figure OAB, and may be said to generate the area OAB. It is convenient to call the area of the figure OAB *positive* if the radius vector OA be positive and swing about O counter-clockwise, and *negative* if it swing clockwise; and this convention conforms to the conventions as to directed lines and angles already in use.



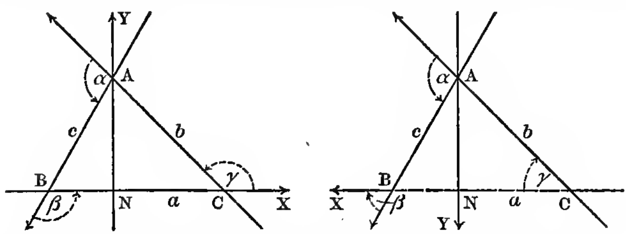
If after generating the area OAB the cord swing back from OB to OA, and its end retrace the same line from B to A, then the area OAB may be thought of as taken up and cancelled, and the sum of the areas OAB, OBA is naught.

So, if C be any point on the line AB, then:

$$\text{area OAB} + \text{area OBC} = \text{area OAC},$$

and $$\text{area OAB} + \text{area OBC} + \text{area OCA} = 0.$$

THEOR. 3. *If* ABC *be an ideal triangle whose sides are a, b, c, and exterior angles α, β, γ, and if* K *be the area of this triangle,*



then
$$\text{K} = \tfrac{1}{2}ab \cdot \sin\gamma = \tfrac{1}{2}ab \cdot \sin \text{C},$$
$$= \tfrac{1}{2}a^2 \cdot \sin\beta \ \sin\gamma / \sin\alpha = \tfrac{1}{2}a^2 \cdot \sin \text{B} \ \sin \text{C} / \sin \text{A},$$
$$= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

For draw NA normal to BC,

then $\because$ $K = \frac{1}{2}\,BC \cdot NA$,

and $NA = CA \sin\gamma$,

$\therefore K = \frac{1}{2} BC \cdot CA \sin\gamma$,

i.e. $K = \frac{1}{2}ab \sin\gamma = \frac{1}{2}ab \sin C$. Q.E.D.

So, $\because b = a \sin\beta/\sin\alpha = a \sin B/\sin A$,

$\therefore K = \frac{1}{2}a^2 \sin\beta \sin\gamma/\sin\alpha = \frac{1}{2}a^2 \sin B \sin C/\sin A$. Q.E.D.

So, $\because \sin\gamma = 2\sin\frac{1}{2}\gamma \cos\frac{1}{2}\gamma$

$= \sqrt{s(s-a)(s-b)(s-c)}/ab$,

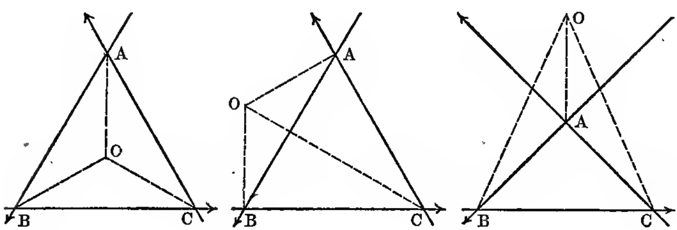
$\therefore K = \sqrt{s(s-a)(s-b)(s-c)}$. Q.E.D.

COR. 1. *If* ABC *be an ideal triangle,* O *any point, and* K *the area of* ABC, *then :*

$$ABC = OAB + OBC + OCA,$$

$$K = \tfrac{1}{2}[OA \cdot OB \sin AOB + OB \cdot OC \sin BOC + OC \cdot OA \sin COA].$$

(*a*) O *within* ABC.



For the three geometric triangles OAB, OBC, OCA are together equal to ABC, as in the first figure, and their areas are all positive.

(*b*) O *without* ABC.

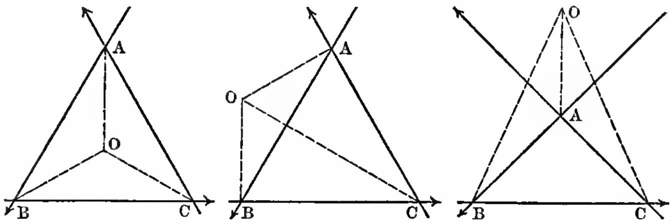
For $\because$ when two of the triangles OAB, OBC, OCA are added and the third is taken away, the triangle ABC remains as in the second figure,

or when from one of them the other two are taken away, it remains as in the third figure,

and while the areas of the two are positive or negative, the third is negative or positive,

∴ the algebraic sum of the areas of these three triangles is that of ABC, in both cases;

$\therefore\ \mathrm{K} = \frac{1}{2}[\mathrm{OA} \cdot \mathrm{OB} \sin \mathrm{AOB} + \mathrm{OB} \cdot \mathrm{OC} \sin \mathrm{BOC} + \mathrm{OC} \cdot \mathrm{OA} \sin \mathrm{COA}]$.



COR. 2. *If* ABC···L *be any polygon,* O *any point in the plane of the polygon, and* K *the area, then:*

$$\mathrm{K} = \tfrac{1}{2}(\mathrm{OA} \cdot \mathrm{OB}\ \sin \mathrm{AOB} + \mathrm{OB} \cdot \mathrm{OC}\ \sin \mathrm{BOC} + \cdots + \mathrm{OL} \cdot \mathrm{OA}\ \sin \mathrm{LOA}).$$

In the three theorems that follow, it is assumed that every motion of a point is the limit of some motion made up of small translations along successive lines, and every motion of a line is the limit of some motion made up of small rotations about successive points.

Either motion is that of a point and a line through it, such that the point always slides along the line, while the line always swings about the point.

E.g. if a line roll round a circle, without sliding upon it, the line always swings about the point of contact, while the point of contact always slides along the tangent line.

The *area swept over by a segment* of a straight line is the algebraic sum of the areas of all the infinitesimal quadrilaterals and triangles passed over, from instant to instant, by the segment.

THEOR. 4. *If* PQR ⋯ TP *be any closed figure traced by the end of a radius vector, drawn from* O, *and varying in length if need be, the area of this figure is the area swept over by the radius vector, and is positive when the bounding line is traced in the positive direction of revolution, and negative when traced in the negative direction.*

(*a*) *No reversals of motion of the vector,* as in the first figure, or *only one reversal,* as in the second figure :

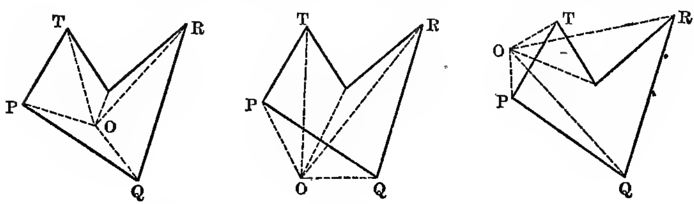
For ∵ there are no intermediate reversals, [hyp.

∴ the figure enclosed by the boundary is swept over once, and but once, by the vector, when it swings in the direction in which the path is traced ;

and ∵ all other figures swept over by the vector in one direction are also swept over in the other direction, and cancelled,

∴ the algebraic sum of the areas of all the figures swept over is the area of the figure enclosed by the boundary,

and this area is positive when the path is traced in the positive direction of rotation, and negative when it is traced in the negative direction. Q.E.D.



(*b*) *Intermediate reversals of motion,* as in the third figure :

For ∵ intermediate reversals occur in consecutive pairs in opposite directions,

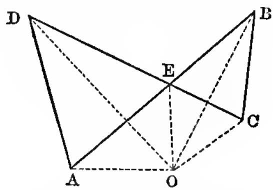
∴ if a point within the enclosure be swept over more than once, it is swept over an odd number of times so as to give an excess of just one passage in the forward direction ;

and ∵ every point without the enclosure is swept over, if at all, the same number of times in each direction, so that any outside area that may be generated is cancelled,

∴ the algebraic sum of the areas of all the figures swept over is the area sought. Q.E.D.

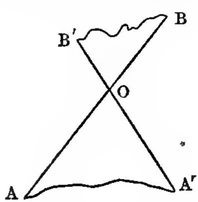
NOTE 1. If the boundary cross itself, the figure is thus divided into two or more parts: the area of each part may be considered separately, and the area of the whole is the algebraic sum of the areas of the several parts.

E.g. the area of the crossed quadrilateral ABCD is the algebraic sum of the areas of the positive triangle AED and the negative angle EBC, and has the sign of the larger.



NOTE 2. In adding two areas any common boundary traversed in opposite directions may be erased.

COR. *If a segment* AB *of a vector* OB *swing about* O *as centre into the position* A'B', *the area of the figure swept over by this segment is the area of the figure* ABB'A', *bounded by the initial and terminal positions of the segment and the paths of its ends.*



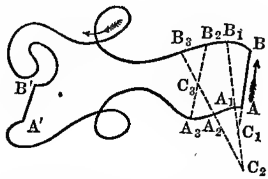
For ∵ AB = OB − OA,

∴ the area K of the figure swept over by the segment AB is the area of the figure swept over by vector OB less the area of the figure swept over by vector OA,

$\therefore$ K $=$ OBB$'$ $-$ OAA$'$ $=$ OBB$'$ $+$ OA$'$A $=$ ABB$'$A$'$A. Q.E.D. [th. 4.

THEOR. 5. *If two points* A, B *move* (forward or backward in any way) *along any paths* AA$'$, BB$'$ *to* A$'$, B$'$, *then the area swept over by the straight line* AB (varying in length if need be) *is the area of the figure* ABB$'$A$'$.

For let the motion of the generator AB be made up of infinitesimal rotations about successive instantaneous centres C_1, C_2, C_3 $\cdots$;

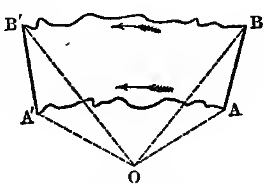


then $\because$ AB sweeps over figures ABB_1A_1, $A_1B_1B_2A_2 \cdots$, [th. 4, cor.

and $\because$ all the intermediate positions A_1B_1, $A_2B_2 \cdots$ of AB are common boundaries of these figures traced in opposite directions,

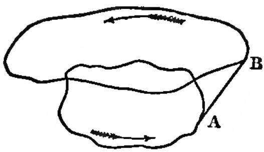
$\therefore$ the sum of all the areas swept over is the area of the figure bounded by the path ABB$'$A$'$A. [th. 4, nt. 2.

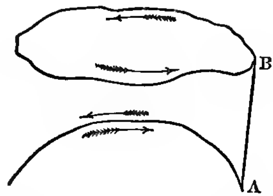
COR. 1. *The area swept over by any straight line* AB *is the sum of the excess of the area of the figure subtended* (from any origin) *by the path of the terminal point* B *over that subtended by the path of the initial point* A *and the excess of the area of the triangle subtended by the initial line* AB *over that subtended by the terminal line* A$'$B$'$;



i.e. ABB$'$A$'$A $=$ (OBB$'$ $-$ OAA$'$) $+$ (OAB $-$ OA$'$B$'$).

Cor. 2. *If the generator* AB *return to its initial position, the area swept over is the excess of the area of the figure bounded by the path of the terminal point* B *over that of the figure bounded by the path of the initial point* A.



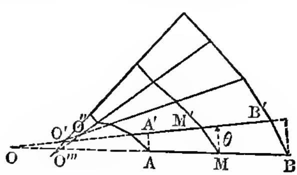


Cor. 3. *If the generator* AB *return to its initial position, and the initial point* A *trace out the same path, to and fro, then the area swept over is the area of the figure bounded by the path of the terminal point* B.

Theor. 6. *If a wheel be affixed to its axis at the mid-point, and if this wheel roll and slide in any way upon a plane while its axis remains parallel to the plane, the area swept over by the axis is the product of its length into the distance rolled by the wheel.*

For, let AB be the axis and M the mid-point ;

let the axis turn about an instantaneous centre O, through an infinitesimal angle θ, and at the same time let the axis slide along its line an infinitesimal distance, to A′B′;



then $\because$ $OA \doteq OA'$, $OB \doteq OB'$, $OM \doteq OM'$, $\sin\theta \doteq \theta$,

$$\therefore \text{area } ABB'A' = OBB' - OAA' \doteq \tfrac{1}{2}(OB^2 - OA^2)\cdot\theta$$

$$= \tfrac{1}{2}(OB - OA)(OB + OA)\cdot\theta = AB\cdot OM\cdot\theta$$

$$= AB\cdot \text{the distance rolled by the wheel at } M,$$

∴ the area swept over by any number of such successive rotations is the product of AB by the distance rolled by the wheel at M. Q.E.D.

COR. 1. *If the wheel be affixed to its axis at any other point,* C, *and the axis turn through an angle,* α, *between its first and last positions, the area swept over is*

$$\text{AB} \cdot \textit{the distance rolled by the wheel at } \text{C} + \text{AB} \cdot \text{CM} \cdot \alpha.$$

For ∵ in the infinitesimal rotation above,

$$\text{area AB} \cdot \text{OM} \cdot \theta = \text{AB} \cdot (\text{OC} + \text{CM}) \cdot \theta$$

$$= \text{AB} \cdot \text{the distance rolled by the wheel at C} + \text{AB} \cdot \text{CM} \cdot \theta,$$

∴ the sum of all such rotations is AB · the distance rolled by the wheel at C + AB · CM · α. [$\alpha = \theta + \theta' + \cdots$.

COR. 2. *If the axis return to its first position without making a complete revolution, the area swept over is* AB · *the distance rolled by the wheel affixed at any point* C. [$\alpha = 0$.

QUESTIONS.

1. If A, B, C be fixed points on a line that turns in a plane through an angle α,

then BC area AB − AB area BC $= \frac{1}{2}$AB · BC · CA · α.

2. If the line in ex. 1 return to its first position:

(*a*) without making a complete revolution,

area B = (AB area C + BC area A) : AC;

(*b*) after making a complete revolution,

area B + π · AB · BC = (AB area C + BC area A) : AC.

3. If the chord AC, in ex. 1, slide round an oval, the area between the oval and the path of B is π · AB · BC.

4. Find the area of the curve traced by a point on the connecting rod of a piston and crank in one revolution; also the distance a small wheel attached at the same point would roll if a plane surface pressed against it.

AMSLER'S PLANIMETER.

Let the axis AB, above noted, be pivoted at A to an arm OA of fixed length that turns about a fixed centre O, so that A traces a fixed circle while B traces any path whatever;

let the wheel be affixed to AB at any point C, but let it be impossible for AB to sweep past OA so that AB, OA can take but one position for one position of B, and, if A encircle O, AB also encircles O in the same direction:

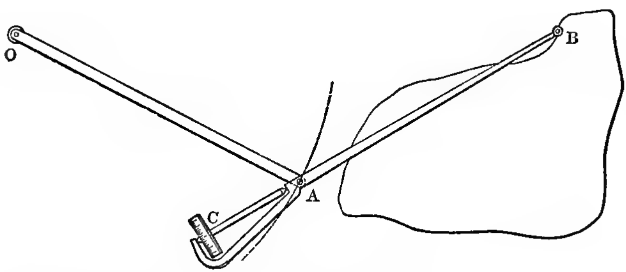
1. If A return to its first position without encircling O,

then ∵ A traces out the same path, to and fro,

∴ the area encircled by B is the area swept over by AB, [theor. 5, cor. 3.

i.e. the area is the product of the number of turns of the wheel into the constant area $2\pi r \cdot AB$, [theor. 6, cor. 2.

wherein r is the radius of the wheel.



2. If A encircle O counter-clockwise,

then the area encircled by B is the area swept over by AB + the area of the circle OA, [theor. 5, cor. 2.

i.e. the area encircled by B is $2\pi r \cdot AB \cdot$ the number of turns of the wheel (positive or negative) $+ AB \cdot CM \cdot \alpha + \pi \cdot OA^2$,

wherein α is 2π. [theor. 6, cor. 1.

The *constants of the planimeter* $2\pi r \cdot AB$, $\pi(2AB \cdot CM + OA^2)$ can be found once for all. The first is the area due to one turn of the wheel; the second is that due to the swinging of the arms OA, AB about O.

§ 6. INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES.

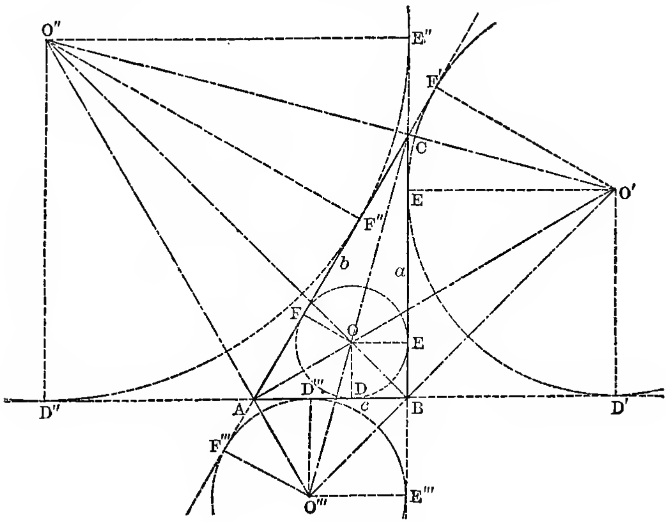
PROB. 2. TO FIND THE RADII OF THE CIRCLES INSCRIBED IN, ESCRIBED, AND CIRCUMSCRIBED ABOUT, ANY TRIANGLE.

For the radius of the inscribed circle, divide the area by half the perimeter.

For the radius of an escribed circle, divide the area by half the perimeter less the side beyond which the circle lies.

For the radius of the circumscribed circle, divide half of either side by the sine of the opposite angle.

For, let ABC be any triangle, and let $r \equiv$ radius of inscribed circle, $r', r'', r''' \equiv$ radii of escribed circles whose centres are O′, O″, O‴, and $R \equiv$ radius of circumscribed circle;



then $\because K = \frac{1}{2} r\,(a+b+c) = rs$, [geom.

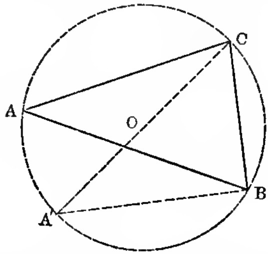
$\therefore r = K/s.$ Q. E. D.

So, $\because K = \frac{1}{2} r'\,(-a+b+c) = r'\,(s-a)$, [geom.

$\therefore r' = K/(s-a)$; and so for r'', r'''. Q. E. D.

Checks: $1/r = 1/r' + 1/r'' + 1/r'''$, $K^2 = r \cdot r' \cdot r'' \cdot r'''$.

About ABC draw a circle and draw CA′, a diameter; join A′B;



then ∵ A = A′, and angle ABC is a right angle, [geom.

and CA′ = a/sin A′ = a/sin A,

∴ R = $\frac{1}{2}a$/sin A ⋯. Q.E.D.

NOTE. a/sin A = b/sin B = c/sin C = 2R.

QUESTIONS.

Find the radii of the inscribed, escribed, and circumscribed circles, and check the work, given:

1. a, 12.7; b, 22.8; c, 51.5.
2. A, 64° 19′; B, 100° 2′; c, 51.25.
3. a, 136; b, 95.2; C, 11° 37′.
4. In a right triangle, 2R + r = s.
5. If R = 2r, the triangle is equilateral.
6. In the ambiguous case the two values of R are equal.
7. The distances from the centre of the inscribed circle to the centres of the three escribed circles are equal to

4R sin $\frac{1}{2}$A ⋯, and to a sec $\frac{1}{2}$A ⋯.

8. The square of the distance between the centres of the inscribed and circumscribed circles is R² − 2Rr.

Prove the equations:

9. $r = (s - a) \tan \frac{1}{2}\mathrm{A}$.
10. $r = s \tan \frac{1}{2}\mathrm{A} \tan \frac{1}{2}\mathrm{B} \tan \frac{1}{2}\mathrm{C}$.
11. $r = a/(\cot \frac{1}{2}\mathrm{B} + \cot \frac{1}{2}\mathrm{C})$, $r' = a/(\tan \frac{1}{2}\mathrm{B} + \tan \frac{1}{2}\mathrm{C})$.

Prove the equations:

12. $R = abc/4K$.

13. $R = s/(\sin A + \sin B + \sin C)$.

14. $r' + r'' + r''' - r = 4R$; $\quad rr'/r''r''' = \tan^2 \frac{1}{2}A$.

15. $K = 4Rr \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

16. $R + r = R(\cos A + \cos B + \cos C)$.

17. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$.

18. In the figure on page 85 CO''' is perpendicular to $O'O''$, AO' to $O''O'''$, $\quad BO''$ to $O'''O'$.

The point O, the co-point of these three perpendiculars, is the *orthocentre* of the triangle $O'O''O'''$.

The triangle ABC, whose sides join the feet of the perpendiculars two and two, is the *pedal triangle* of $O'O''O'''$.

19. The circle circumscribed about ABC passes through the mid-points of the triangle $O'O''O'''$, and through the mid-points of the segments OO', OO'', OO'''.

This circle is the *nine-point circle* of the triangle $O'O''O'''$.

20. The nine-point circle of a triangle circumscribes its pedal triangle, passes through the mid-point of each side, and bisects the lines joining the vertices to the orthocentre.

21. If a, b, c be the sides of a triangle, and ρ be the radius of the circle inscribed in a triangle whose sides are $b+c$, $c+a$, $a+b$, then $\quad \rho^2 = 2Rr$.

22. If a, b, c be the sides of a triangle, and m, n, p be the altitudes, then $\quad mnp = (a+b+c)^3 r^3/abc$.

23. If u, v, w be the distances between the excentres of a triangle, then $\quad uvw \sin A \sin B \sin C = 8r'r''r'''$.

24. Find the radii of the circles that touch two sides of a triangle and the inscribed circle.

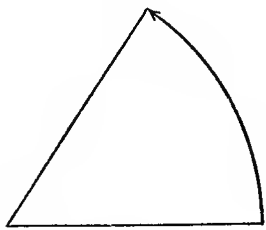
So, of those that touch the circumscribed circle.

25. Find the relation which exists between the angles of a triangle whose orthocentre lies on the inscribed circle.

IV. DERIVATIVES, SERIES, AND TABLES.

§ 1. CIRCULAR MEASURE OF ANGLES.

In the applications of trigonometry to numerical problems, *e.g.* the solution of triangles, the most convenient unit of angular measure is the right angle, or the degree, the ninetieth part of a right angle; but in certain other problems, *e.g.* the computation of trigonometric ratios and their logarithms, that angle which lies at the centre of a circle, and whose bounding arc is just as long as the radius of the circle, is a better unit. This unit angle is called a *radian*, and its magnitude is independent of the length of the radius. [geom.



Radians may be indicated by the sign $^{(r)}$ just as degrees, minutes, and seconds are indicated by the signs $^\circ$, $'$, $''$; and since the ratio of the half circumference of a circle to its radius is π, [3.141592 $\cdots$] and angles at the centre are proportional to their arcs, two right angles are equal to π radians.

The primary equation expressing the relation between degrees and radians is $\pi^{(r)} = 180^\circ$: from this it follows that

$$\tfrac{1}{2}\pi^{(r)} = 90^\circ, \quad \tfrac{1}{4}\pi^{(r)} = 45^\circ, \quad \tfrac{1}{6}\pi^{(r)} = 30^\circ, \cdots,$$

$$1^{(r)} = 180^\circ/\pi = 57^\circ\, 17'\, 44.8'',$$

$$1^\circ = \pi^r/180 = .0174533^r, \quad 1' = .0002909^r, \cdots,$$

and the measure of other angles is expressed by the ratio of the bounding arc to the radius.

QUESTIONS.

1. Prove that the number of radians in an angle is expressed by the ratio of the arc subtending it to the radius of the circle, *i.e.* by the number of radii in the arc.

2. Express in degree-measure the angles:

$\frac{1}{5}\pi$, $\frac{1}{7}\pi$, $\frac{1}{9}\pi$, $\frac{5}{3}\pi$, 3.1416^r, $.7854^r$, 1^r, 1.5^r, -2^r, $(\pi+1)^r$.

3. Express in radius-measure the angles:

$14°$, $15°$, $24°$, $120°$, $137° 15'$, $-4800°$, $13'$, $24''$.

4. If the radius be an inch, find the length of the arcs:

$14°$, $15°$, $120°$, $57° 17' 44.8''$, $1°$, $\frac{1}{5}\pi$, $\frac{1}{7}\pi$, 2^r, $\pi+1^r$.

So, if the radius be five inches.

5. How many radii in an arc of: $20°$, $180°$, 3^r?

6. If the radius be 10 inches, find the number of radians subtended by an arc of: 13 inches, π inches, $10°$, $5' 13''$, three quadrants.

7. The angle 3.42^r is subtended by an arc of 5.71 inches: find the radius; the arcs opposite $\frac{1}{2}\pi^r$, 1^r, $5°$; the angles in radians and in degrees opposite a one-inch arc, a two-radius arc, five quadrants.

8. If the circumference of a circle be 30 inches, find the arcs opposite π^r, $30°$, 3^r.

9. How many radians and how many degrees are subtended by: $2\frac{1}{2}$ radius arcs, π radius arcs, $3\frac{1}{2}$ quadrants?

10. How many radians in $17° 13' 15''$? in $10°$?

11. An angle of three radians at the centre of a sphere subtends a two-foot arc of a great circle: find the radius.

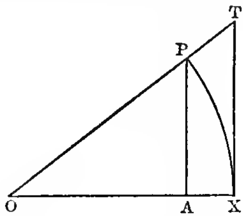
12. The apparent diameter of the sun, as seen from the earth, is half a degree; a planet crosses the sun's disk in a straight line at a distance from its centre equal to three-fifths of the sun's diameter: show that the angle subtended at the earth by the part of the planet's path projected on the sun is $\pi^r/450$.

§ 2. DERIVATIVES OF TRIGONOMETRIC RATIOS.

THEOR. 1. *If θ be the circular measure of a positive acute angle, then* $\sin\theta < \theta < \tan\theta$.

For, let XOP be any positive acute angle; with O as centre, and any radius OX, describe a circle cutting OP in P;

through P, X draw normals to OX, cutting OX in A and OP in T;



then $\because$ AP < XP < XT, [geom.

and $AP/OX \equiv \sin\theta$, $XP/OX \equiv \theta$, $XT/OX \equiv \tan\theta$,

$\therefore \sin\theta < \theta < \tan\theta$. Q.E.D.

COR. 1. *If θ approach zero, the ratios $\theta/\sin\theta$, $\theta/\tan\theta$ approach unity.*

For $\because \sin\theta < \theta < \tan\theta$, [above.

$\therefore 1 < \theta/\sin\theta < \sec\theta$, [div. by $\sin\theta$.

and $\cos\theta < \theta/\tan\theta < 1$; [mult. by $\cos\theta$.

and $\because \cos\theta \doteq 1$, and $\sec\theta \doteq 1$, when $\theta \doteq 0$,

$\therefore \theta/\sin\theta \doteq 1$, and $\theta/\tan\theta \doteq 1$, when $\theta \doteq 0$.

For definition of limit, derivative, etc., and for proof of the necessary properties of limits and derivatives, see any good work on the differential calculus.

Some of the fundamental properties of derivatives are, for convenience of reference, set down here as lemmas without proof; they are given in two forms:

If U, V *be functions of any variable x, then:*

LEM. 1. $D_x(U+V) = D_xU + D_xV$,

$$\delta(U+V) \doteq \delta U + \delta V.$$

LEM. 2. $D_x(U \cdot V) = V \cdot D_xU + U \cdot D_xV$,

$$\delta(U \cdot V) \doteq V \cdot \delta U + U \cdot \delta V.$$

LEM. 3. $D_x(U/V) = (V \cdot D_xU - U \cdot D_xV)/V^2$,

$$\delta(U/V) \doteq (V \cdot \delta U - U \cdot \delta V)/V^2.$$

LEM. 4. $D_xU^n = nU^{n-1} \cdot D_xU$, $\quad \delta U^n \doteq nU^{n-1} \cdot \delta U$.

LEM. 5. $D_x \log_e U = D_xU/U$, $\quad \delta \log_e U = \delta U/U$.

LEM. 6. *If* U *be a function of* V, *and* V *a function of x, then:*

$$D_xU = D_vU \cdot D_xV.$$

wherein $D_x \equiv$ *x-derivative of*, $\delta \equiv$ *a very small increment of*, and the sign $\doteq$, read *approaches*, means that the difference of the two members is infinitesimal as to either of them.

DERIVATIVES OF THE RATIOS.

THEOR. 2. *If θ be the circular measure of any plane angle, then:*

$$D_\theta \sin\theta = \cos\theta, \qquad D_\theta \csc\theta = -\cot\theta \csc\theta,$$
$$D_\theta \cos\theta = -\sin\theta, \qquad D_\theta \sec\theta = \tan\theta \sec\theta,$$
$$D_\theta \tan\theta = \sec^2\theta, \qquad D_\theta \cot\theta = -\csc^2\theta.$$

For, let θ' be an infinitesimal angle, the increment of θ;

then $\because \sin(\theta+\theta') - \sin\theta = 2\cos(\theta+\frac{1}{2}\theta')\sin\frac{1}{2}\theta'$,

$\therefore [\sin(\theta+\theta') - \sin\theta]/\theta = \cos(\theta+\frac{1}{2}\theta') \cdot \sin\frac{1}{2}\theta/\frac{1}{2}\theta'$.

But $\because$ θ' is the increment of θ, [hyp.

and $\sin(\theta+\theta') - \sin\theta$ is the consequent increment of $\sin\theta$,

$\therefore \lim(\text{inc}\sin\theta/\text{inc}\,\theta), \equiv D_\theta\sin\theta, = \cos\theta$. Q.E.D. [th. 1.

So, $\because \cos(\theta+\theta') - \cos\theta = -2\sin(\theta+\frac{1}{2}\theta')\sin\frac{1}{2}\theta'$,

$\therefore [\cos(\theta+\theta') - \cos\theta]/\theta' = -\sin(\theta+\frac{1}{2}\theta') \cdot \sin\frac{1}{2}\theta'/\frac{1}{2}\theta'$,

$\therefore \lim(\text{inc}\cos\theta/\text{inc}\,\theta), \equiv D_\theta\cos\theta, = -\sin\theta$. Q.E.D. [th. 1.

So, $D_\theta\tan\theta, \equiv D_\theta(\sin\theta/\cos\theta)$,

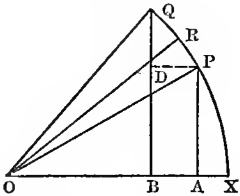
$= (\cos\theta\, D_\theta\sin\theta - \sin\theta\, D_\theta\cos\theta)/\cos^2\theta$

$= (\cos^2\theta + \sin^2\theta)/\cos^2\theta = \sec^2\theta$.

So for $D_\theta \csc\theta = D_\theta(1/\sin\theta)$, for $D_\theta \sec\theta$, for $D_\theta \cot\theta$

GEOMETRIC PROOF.

Let O-XP be any circle, and Q a point on this circle near P;
bisect arc PQ at R, and join OX, OP, OQ, OR;
draw AP, BQ normal to OX;
join P, Q, and through P draw a parallel to OX meeting BQ in D;
let $\theta \equiv \angle$XOP, $\theta' \equiv \angle$POQ, $(\theta+\frac{1}{2}\theta') \equiv \angle$XOR, $r \equiv$ radius of circle;



then $\because$ $\sin\theta = \text{AP}/r$, $\sin(\theta+\theta') = \text{BQ}/r$, $\theta' = \text{arc PQ}/r$,
and $\angle\text{DQP} = \angle\text{XOR}$, [geom.

$$\therefore [\sin(\theta+\theta') - \sin\theta]/\theta' = \text{DQ}/\text{arc PQ}$$
$$= (\text{DQ}/\text{chord PQ}) \cdot (\text{chord PQ}/\text{arc PQ})$$
$$= \cos(\theta+\tfrac{1}{2}\theta') \cdot (\text{chord PQ}/\text{arc PQ}),$$

$\therefore \lim(\text{inc}\sin\theta/\text{inc}\,\theta), \equiv D_\theta \sin\theta, = \cos\theta$; Q.E.D. [th. 1.
and so for $D_\theta \cos\theta$, $D_\theta \tan\theta \cdots$.

DERIVATIVES OF ANTIFUNCTIONS.

THEOR. 3. $D_x\, sin^{-1}x = 1/\sqrt{(1-x^2)}$,
$D_x\, cos^{-1}x = -1/\sqrt{(1-x^2)}$,
$D_x\, tan^{-1}x = 1/(1+x^2)$,
$D_x\, cot^{-1}x = -1/(1+x^2)$,
$D_x\, sec^{-1}x = 1/x\sqrt{(x^2-1)}$,
$D_x\, csc^{-1}x = -1/x\sqrt{(x^2-1)}$.

For let $\theta = \sin^{-1}x$;
then $\because$ $\sin\theta = x$,

$\therefore D_x \sin\theta, = \cos\theta \cdot D_x\theta, = 1$,

$\therefore D_x\theta = 1/\cos\theta = 1/\sqrt{(1-x^2)}$; Q.E.D.

and so for the rest.

NOTE. When x stands for $\sin\theta$ or $\cos\theta$, x may have any value positive or negative not larger than unity; when x stands for $\tan\theta$ or $\cot\theta$, x may have any value whatever; and when x stands for $\sec\theta$ or $\csc\theta$, x may have any value not smaller than unity: for if, in the formulæ above, x exceeds the bounds named, the function is imaginary.

QUESTIONS.

1. If θ be any plane angle and θ' be the increment of θ, then:

$$\text{inc}^2 \sin\theta = -(2\sin\tfrac{1}{2}\theta')^2 \sin(\theta+\theta'),$$
$$\text{inc}^2 \cos\theta = -(2\sin\tfrac{1}{2}\theta')^2 \cos(\theta+\theta'),$$
$$\text{inc}^4 \sin\theta = \quad (2\sin\tfrac{1}{2}\theta')^4 \sin(\theta+2\theta'),$$
$$\text{inc}^4 \cos\theta = \quad (2\sin\tfrac{1}{2}\theta')^4 \cos(\theta+2\theta'),$$

wherein $\text{inc}^2\sin\theta \equiv$ the increment of the increment of $\sin\theta$,

i.e. $\quad [\sin(\theta+2\theta')-\sin(\theta+\theta')]-[\sin(\theta+\theta')-\sin\theta]$,

or $\quad \sin(\theta+2\theta')-2\sin(\theta+\theta')+\sin\theta$;

and $\quad \text{inc}^4\sin\theta \equiv \text{inc inc inc inc} \sin\theta$,

i.e. $\quad \sin(\theta+4\theta')-4\sin(\theta+3\theta')+6\sin(\theta+2\theta')$
$\qquad -4\sin(\theta+\theta')+\sin\theta$.

2. If δa, δb, δc, δA, δB be any simultaneous small changes in the values of a, b, c, A, B, that are consistent with the known relations of the parts of a right triangle

$$[A+B=90^\circ,\quad a^2+b^2=c^2,\quad a=c\sin A,\quad b=c\cos a],$$

then $\quad \delta B=-\delta A,\quad \delta c \doteq a/c\cdot\delta a+b/c\cdot\delta b=\sin A\cdot\delta a+\cos A\cdot\delta b$,

$$\delta a \doteq \sin A\cdot\delta c+c\cos A\cdot\delta A,\qquad \delta b \doteq \cos A\cdot\delta c-c\sin A\cdot\delta A,$$

and [eliminate δc from the last two equations]

$$\delta A \doteq \cos A/c\cdot\delta a-\sin A/c\cdot\delta b=(b\delta a-a\delta b)/(a^2+b^2).$$

3. If, in a right triangle, only the values of a, b be given, and if these have the possible errors $\pm a'$, $\pm b'$; *i.e.* if a may possibly differ from its assumed value by either $+a'$ or $-a'$, and b by either $+b'$ or $-b'$; show from ex. 2 that the resulting values of c, A will have the possible errors

$$\pm(aa'+bb')/c = \pm(a'\sin A+b'\cos A), \qquad [a', b' \text{ positive.}$$

and $\quad \pm(ab'+ba')/c^2 = \pm(b'\sin A+a'\cos A)/c$.

So, if only b, c be given, with the possible errors $\pm b'$, $\pm c'$, find the possible errors of the other sides and angles.

So, if only b, A be given, or only c, A, with the possible errors $\pm b'$, $\pm$A$'$, or $\pm c'$, $\pm$A$'$.

4. From the known relations of the parts of an oblique triangle [$A+B+C=180°$, $a \sin B = b \sin A, \cdots$] prove that

(*a*) $\delta A + \delta B + \delta C = 0$,

(*b*) $b \cos A \cdot \delta A - a \cos B \cdot \delta B - \sin B \cdot \delta a + \sin A \cdot \delta b = 0$,

$c \cos B \cdot \delta B - b \cos C \cdot \delta C - \sin C \cdot \delta b + \sin B \cdot \delta c = 0$,

$a \cos C \cdot \delta C - c \cos A \cdot \delta A - \sin A \cdot \delta c + \sin C \cdot \delta a = 0$.

From these equations, by elimination and reduction, derive

(*c*) $b \cdot \delta C + c \cos A \cdot \delta B - \sin A \cdot \delta c + \sin C \cdot \delta a = 0$,

$c \cdot B\delta + b \cos A \cdot \delta C - \sin A \cdot \delta b + \sin B \cdot \delta a = 0$,

with four symmetric equations; and

(*d*) $b \sin C \cdot \delta A - \delta a + \cos C \cdot \delta b + \cos B \cdot \delta c = 0$,

with two symmetric equations.

5. If in an oblique triangle only a, B, C be given, and if their possible errors be $\pm a/10000$, $\pm 10''$, $\pm 15''$, find the possible errors of A [ex. 4, *a*]; of b [ex. 4, *c*]; of c [ex. 4, *c*].

Find the values of these possible errors when ABC is very nearly equilateral, 5000 feet on each side.

6. Given the values of C, a, b, with the possible errors $\pm$C$'$, $\pm a'$, $\pm b'$, find the possible errors of B, A, c [ex. 4, *c*, *d*].

7. Given A, a, b, with the possible errors $\pm$A$'$, $\pm a'$, $\pm b'$, find the possible errors of B [ex. 4, *b*]; of C, c.

8. Given A, B, b, with possible errors $\pm$A$'$, $\pm$B$'$, $\pm b'$, find the possible errors of C, a, c: first, when, as in all the above cases, the computation is assumed to be exact; second, when C, a, c have the further possible errors $\pm$C$''$, $\pm a''$, $\pm c''$ from decimal figures omitted in the computation.

9. Given a, b, c, with possible errors $\pm a'$, $\pm b'$, $\pm c'$: find the possible error of A, with a possible error in computation of $\pm$A$''$.

§ 3. EXPANSION OF TRIGONOMETRIC RATIOS.

In the expansion of trigonometric ratios the following properties of series are made use of: they are all proved in works on algebra, and are quoted here for convenient reference.

LEM. 7. *If, after a given term, the terms of a series form a decreasing geometric progression, the series is convergent.*

LEM. 8. *If one series be convergent, and if the terms of another series be not larger than the corresponding terms of the first series, the second series is convergent.*

LEM. 9. *If, after a given term, the ratio of each term of a series to the term before it be smaller than some fixed number that is itself smaller than unity, the series is convergent.*

COR. *The series* $A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$ *is convergent for all values of* x *that make the limit of the ratio of the* $(n+1)^{th}$ *term to the* n^{th} *term smaller than unity when* n *becomes very great.*

LEM. 10. *If in the series* $A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$, *the limit of the ratio of the* $(n+1)^{th}$ *term to the* n^{th} *term, for any value of* x, *be smaller than unity, then, in the derivative series* $A_1 + 2A_2x + 3A_3x^2 + \cdots$, *the limit of the* $(n+1)^{th}$ *term to the* n^{th} *term, for this value of* x, *is smaller than unity, and this series is convergent.*

LEM. 11. *If two series arranged to rising powers of any same variable be equal for all values of the variable that make them both convergent, the coefficients of like powers of the variable are equal.*

THEOR. 4. *If* θ *be the circular measure of any plane angle, then:*

$$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \cdots,$$
$$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \cdots.$$

For assume $\sin\theta = A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3 + \cdots$, wherein the A's are unknown but constant, and θ has such values as make the series convergent,

and find the first two θ-derivatives of both members of the equation;

then $\cos\theta = A_1 + 2A_2\theta + 3A_3\theta^2 + \cdots$,

and $-\sin\theta = 2A_2 + 2\cdot 3A_3\theta + \cdots$,

i.e. $\sin\theta = -2A_2 - 2\cdot 3A_3\theta$;

and both of these derivative series are convergent. [lem 9, cor.

Take 0 as one of the values of θ;

then $\because \sin 0 = 0$ and $A_0 + A_1 0 + A_2 0^2 + A_3 0^3 + \cdots = A_0$,

$\therefore A_0 = 0.$

So, $\because \cos 0 = 1$ and $A_1 + 2A_2 0 + 3A_3 0^2 + \cdots = A_1$,

$\therefore A_1 = 1.$

So, $\because A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3 + \cdots = -2A_2 - 2\cdot 3A_3\theta - 3\cdot 4A_4\theta^2 - \cdots$ for all values of θ that make both series convergent,

$\therefore A_0 = -2A_2, \quad A_2 = 3\cdot 4A_4, \quad A_4 = 5\cdot 6A_6 \cdots,$

and $A_1 = -2\cdot 3A_3, \quad A_3 = -4\cdot 5A_5, \quad A_5 = -6\cdot 7A_7 \cdots$; [lem. 11.

$\therefore A_0, A_2, A_4, A_6, A_8, \cdots = 0,$

and $A_1 = 1, \quad A_3 = -1/3!, \quad A_5 = 1/5!, \quad A_7 = -1/7 \cdots$;

$\therefore \sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \cdots,$

and $\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \cdots.$

NOTE. These series are convergent for all finite values of θ. For the ratios of successive terms, in that for the sine, are

$$\theta^2/(2\cdot 3), \quad \theta^2/(4\cdot 5), \quad \theta^2/(6\cdot 7) \cdots;$$

and, in that for the cosine,

$$\theta^2/(1\cdot 2), \quad \theta^2/(3\cdot 4), \quad \theta^2/(5\cdot 6), \cdots;$$

i.e. series of fractions such that the limit of the $(n+1)^{th}$ term to the n^{th} term is smaller than unity whatever be the value of θ.

But they converge rapidly only when θ is quite small.

COR. 1. $\tan\theta = \theta + \theta^3/3 + 2\theta^5/(3\cdot 5) + 17\theta^7/(3^2\cdot 5\cdot 7) + 62\theta^9/(3^4\cdot 5\cdot 7) + \cdots,$

$\cot\theta = 1/\theta - \theta/3 - \theta^3/(3^2\cdot 5) - 2\theta^5/(3^3\cdot 5\cdot 7) - \theta^7/(3^3\cdot 5^2\cdot 7) - \cdots,$

$$\sec\theta = 1 + \theta^2/2 + 5\theta^4/(2^3\cdot 3) + 61\theta^6/(2^4\cdot 3^2\cdot 5)$$
$$+ 277\theta^8/(2^7\cdot 3^2\cdot 7) + \cdots,$$

$$\csc\theta = 1/\theta + \theta/(2\cdot 3) + 7\theta^3/(2^6\cdot 3^2\cdot 5)$$
$$+ 31\theta^5/(2^4\cdot 3^3\cdot 5\cdot 7) + 127\theta^7/(2^7\cdot 3^3\cdot 5\cdot 7) + \cdots.$$

For the tangent, divide the series for the sine by that for the cosine ;

for the cotangent, divide the series for the cosine by that for the sine ;

for the secant, divide unity by the series for the cosine ;

for the cosecant, divide unity by the series for the sine.

NOTE. The series for $\tan\theta$ and $\sec\theta$ are convergent only when $\theta < \frac{1}{2}\pi$, for $\tan\theta$ and $\sec\theta$ are finite and continuous functions of θ for all values of θ smaller than $\frac{1}{2}\pi$; but when $\theta = \frac{1}{2}\pi$ their values are infinite. So, the series for $\theta\cot\theta$ and $\theta\csc\theta$ are convergent only when $\theta < \pi$.

COR. 2. $$\textit{log-sin}\,\theta = \log\theta - \theta^2/(2\cdot 3) - \theta^4/(2^2\cdot 3^2\cdot 5)$$
$$- \theta^6/(3^4\cdot 5\cdot 7) - \cdots,$$

$$\textit{log-cos}\,\theta = -[\theta^2/2 + \theta^4/(2^2\cdot 3) + \theta^6/(3^2\cdot 5)$$
$$+ 17\theta^8/(2^3\cdot 3^2\cdot 5\cdot 7) + \cdots].$$

For $\because \mathrm{D}_\theta$ log-sin $\theta = \cos\theta/\sin\theta = \cot\theta = 1/\theta - \theta/3$
$$- \theta^3/(3^2\cdot 5) - \cdots, \qquad \text{[lem.}$$

$\therefore$ log-sin $\theta = \log\theta - \theta^2/(2\cdot 3) - \theta^4/(2^2\cdot 3^2\cdot 5)$
$$- \theta^6/(3^4\cdot 5\cdot 7) - \cdots, \qquad \text{[lem.}$$

i.e. log-sin θ = the series whose θ-derivative is the above series for $\cot\theta$, and which, as $\theta \doteq 0$, approaches to $\log\theta$ as log-sin θ must do.

So, $\because \mathrm{D}_\theta$log-cos $\theta = -\sin\theta/\cos\theta = -\tan\theta = -(\theta + \theta^3/3 + \cdots)$,

$\therefore$ log-cos $\theta = -[\theta^2/2 + \theta^4/(2^2\cdot 3) + \theta^6/(3^2\cdot 5)$
$$+ 17\theta^8/(2^3\cdot 3^2\cdot 5\cdot 7) + \cdots].$$

NOTE. The series for log-sin θ is convergent for all values of θ smaller than π ; that for log-cos θ for all values smaller than $\frac{1}{2}\pi$.

GREGORY'S THEOREM.

THEOR. 5. *If x be any number smaller than unity, then*
$$tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \tfrac{1}{11}x^{11} + \cdots.$$

For, assume $\tan^{-1}x = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \cdots$,
and take the x-derivative of both members;

then $D_x \tan^{-1}x = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \cdots$;

and $\because D_x \tan^{-1}x = 1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \cdots$, [theor. 3.

$\therefore A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots = 1 - x^2 + x^4 - x^6 + \cdots$,
for all values of x that make both series convergent,

$\therefore A_2, A_4, A_6, \cdots = 0$, [lem. 11.

and $A_1 = 1, \quad A_3 = -1/3, \quad A_5 = 1/5, \quad A_7 = -1/7 \cdots$.

So, take 0 as a value of x,

then $\tan^{-1}0 = A_0 + A_1 0 + A_2 0^2 + A_3 0^3 + \cdots$, and $A_0 = 0$;

$\therefore \tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \cdots$. Q. E. D.

NOTE. This series is convergent when $x \leqslant 1$; but it converges very slowly when x is near one, and rapidly only when x is small.

In the computation of the length of an arc, and so of the circumference of a circle and of π, either of the equations below gives a practical working rule:

$$\tfrac{1}{6}\pi = \tan^{-1} 1/\sqrt{3} = [1 - 1/(3\cdot 3) + 1/(5\cdot 3^2) - 1/(7\cdot 3^3) + 1/(9\cdot 3^4) - 1/(11\cdot 3^5) + 1/(13\cdot 3^6) - \cdots]/\sqrt{3}.$$

$$\begin{aligned}\tfrac{1}{4}\pi &= \tan^{-1} 1/2 + \tan^{-1} 1/3 \\ &= 1/2 - 1/(3\cdot 2^3) + 1/(5\cdot 2^5) - 1/(7\cdot 2^7) + \cdots \\ &\quad + 1/3 - 1/(3\cdot 3^3) + 1/(5\cdot 3^5) - 1/(7\cdot 3^7) + \cdots.\end{aligned}$$

$$\begin{aligned}\tfrac{1}{4}\pi &= 4\tan^{-1} 1/5 - \tan^{-1} 1/239, \\ &= 4\tan^{-1} 1/5 - \tan^{-1} 1/70 + \tan^{-1} 1/99 \\ &= 4[1/5 - 1/(3\cdot 5^3) + 1/(5\cdot 5^5) - 1/(7\cdot 5^7) + \cdots] \\ &\quad - [1/70 - 1/(3\cdot 70^3) + 1/(5\cdot 70^5) - 1/(7\cdot 70^7) + \cdots] \\ &\quad + [1/99 - 1/(3\cdot 99^3) + 1/(5\cdot 99^5) - 1/(7\cdot 99^7) + \cdots].\end{aligned}$$

With the last of these equations, the work of computati
may take this form :

			+	−
5	4.			
25	.8	1	.8	
25	.032	3		.010 666 66
25	.001 28	5	.000 256	
25	.000 051 2	7		.000 007 31
25	.000 002 048	9	.000 000 228	
	.000 000 082	11		.000 000 00
70	1.			
70	.014 285 714	1		.014 285 71
70	.000 204 082			
	.000 002 915	3	.000 000 972	
99	1.			
99	.010 101 010	1	.010 101 010	
99	.000 102 030			
	.000 001 030	3		.000 000 34
			.810 358 210	.024 960 04
			.024 960 045	
			.785 398 165	
			4	

$\therefore$ $\pi = 3.141\,592\,6$ to eight figur

§ 4. COMPUTATION OF TRIGONOMETRIC RATIOS.

PROB. 1. TO COMPUTE A TABLE OF SINES AND COSINES

(*a*) *For angles* $0° \cdots 30°$:

Replace θ by $1'$, $2'$, $3' \cdots$ in the formulæ of theor. 4.

E.g. $\because 1' = \pi/(180 \cdot 60) = 3.141\,592\,653\,589.793/10\,800$

$$= .000\,290\,888\,208\,666,$$

$$\therefore \sin 1' = .000\,290\,888\,208\,666 - .000\,290\,888\,208\,666^3/3\,!$$

$$= .000\,290\,8882;$$

$$\cos 1' = 1 - .000\,290\,888\,208\,666^2/2\,! + \cdots$$

$$= .999\,999\,9577;$$

$$\sin 2' = 2\times .000\,290\,888\,208\,666$$
$$\quad -2^3\times .000\,290\,888\,208\,666^3/3\,! + \cdots$$
$$= .000\,581\,7764;$$
$$\cos 2' = 1 - 2^2\times .000\,290\,888\,208\,666^2/2\,! + \cdots$$
$$= .999\,999\,8308.$$

NOTE 1. The fraction $\pi/10\,800$ once raised to the required powers, first, second, third $\cdots$, and divided by the factorials $1\,!$, $2\,!$, $3\,!\cdots$, thereafter only simple multiples of the quotients are used. [A small table of these powers and quotients, correct to twenty decimal places, is given on page 38 of Jones' Six-place Tables, and a larger table in Callet's Tables de Logarithmes.] At first but two terms of the series are needed; but later, when θ is larger and the series therefore converges less rapidly, and at critical points, *e. g.* the finding of the value of $.485795000\pm$, correct to five figures, more terms must be taken.

E.g. for $30°$, $\theta = \frac{1}{6}\pi = .52360$ nearly;

and $\sin 30° = .52360 - .5236^3/6 + .5236^5/120 - \cdots$

$$= .52360 - .02392 + .00033 - .00000 + \cdots$$

$= .5$, the true value, within less than $.00005$;

i.e. by the use of three terms of the series, the sine is found correct to four decimal places, the same degree of accuracy as that assumed for the value of π.

NOTE 2. The method shown above may serve whether individual ratios be sought or an entire table; but if a table, then the following method may also be used.

Assume $\sin 1'$ as differing insensibly from arc $1'$,

i.e. that $\sin 1' = .000\,290\,8882$,

hence, that $\cos 1', = \sqrt{1-\sin^2 1'}, = .999\,999\,9577$;

then in the formulæ

$$\sin(\theta+\theta') = 2\sin\theta\cos\theta' - \sin(\theta-\theta'),$$ [II, th. 11, cr. 2.

$$\cos(\theta+\theta') = 2\cos\theta\cos\theta' - \cos(\theta-\theta'),$$

replace θ by $1'$, $2'$, $3'\cdots$ in turn, and θ' by $1'$.

E.g. $\sin 2' = 2 \sin 1' \cos 1' - \sin 0'$
$= 2 \times .000\,290\,8882 \times .999\,999\,9577 - 0$
$= .000\,581\,7764 \times (1 - .000\,000\,0423)$
$= .000\,581\,7764$;

and $\sin 3' = 2 \sin 2' \cos 1' - \sin 1'$
$= 2 \times .000\,581\,7764 \times (1 - .000\,000\,0423)$
$- .000\,290\,8882$
$= .000\,872\,6646$.

So, $\cos 2' = 2 \cos 1' \cos 1' - \cos 0'$
$= 2 \times .999\,999\,9577 \times (1 - .000\,000\,0423) - 1$
$= .999\,999\,8308$;

and $\cos 3' = 2 \cos 2' \cos 1' - \cos 1'$
$= 2 \times .999\,999\,8308 \times (1 - .000\,000\,0423)$
$- .999\,999\,9577$
$= .999\,999\,6193$.

(*b*) *For angles* $30° \cdots 45°$:
Replace θ' by $1'$, $2'$, $3' \cdots$ in the formulæ
$\sin(30° + \theta') = \cos\theta' - \sin(30° - \theta')$, [ad. th., $\sin 30° = \frac{1}{2}$.
$\cos(30° + \theta') = \cos(30° - \theta') - \sin\theta'$.

E.g. $\sin 30°\,1' = \cos 1' - \sin 29°\,59'$
$= .999\,999 \cdots - .499\,75 = .500\,25$,

and $\sin 30°\,2' = \cos 2' - \sin 29°\,58'$
$= .999\,999 \cdots - .499\,50 = .500\,50$.

So, $\cos 30°\,1' = \cos 29°\,59' - \sin 1'$
$= .866\,17 - .000\,29 = .865\,88$,

and $\cos 30°\,2' = \cos 29°\,58' - \sin 2' = .865\,73$.

(*c*) *For angles* $45° \cdots 90°$: apply the formulæ
$\sin(45° + \theta') = \cos(45° - \theta')$, [II, theor. 6.
$\cos(45° + \theta') = \sin(45° - \theta')$.

E.g. $\sin 45°\,1' = \cos 44°\,59' = .707\,31$,
$\cos 45°\,1' = \sin 44°\,59' = .706\,90$.

VERIFICATION.

NOTE 3. The results are tested in many ways:

(*a*) $\because \sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1-\cos\theta)}$, $\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1+\cos\theta)}$,

$\therefore$ from $\cos 45°, = \sqrt{\frac{1}{2}}$, are found in succession the sines and cosines of 22° 30′, 11° 15′ ⋯.

So, from $\cos 30°, = \frac{1}{2}\sqrt{3}$, are found in succession the sines and cosines of 15°, 7° 30′ ⋯.

(*b*) $\because \sin 2\theta = 2\sin\theta\cos\theta$, [II, theor. 13.

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, [II, theor. 15, cor.

and $\sin 36° = \cos 54°$, [II, theor. 6.

$\therefore 2\sin 18°\cos 18° = 4\cos^3 18° - 3\cos 18°$,

$\therefore 2\sin 18° = 4(1-\sin^2 18°) - 3$,

$\therefore \sin 18° = \frac{1}{4}(\sqrt{5}-1)$, $\cos 18° = \frac{1}{4}\sqrt{(10+2\sqrt{5})}$;

thence, in turn, the sines and cosines of 9°, 4° 30′, 2° 15′ ⋯.

(*c*) From $\cos 36°, = \cos^2 18° - \sin^2 18° = \frac{1}{4}(\sqrt{5}+1)$,

and $\sin 36°, = \sqrt{(1-\cos^2 36°)} = \frac{1}{4}\sqrt{(10-2\sqrt{5})}$,

are found the sine and cosine of (36° − 30°), *i.e.* of 6°, thence in turn the sine and cosine of 3°, 1° 30′, 45′, ⋯.

(*d*) From $\sin(36°+\theta') - \sin(36°-\theta'), = 2\cos 36°\sin\theta'$
$= \frac{1}{2}(\sqrt{5}+1)\sin\theta'$,

subtract $\sin(72°+\theta') - \sin(72°-\theta'), = 2\cos 72°\sin\theta'$
$= \frac{1}{2}(\sqrt{5}-1)\sin\theta'$;

then, $\sin(36°+\theta') - \sin(36°-\theta')$

$= \sin(72°+\theta') - \sin(72°-\theta') + \sin\theta'$:

a formula that serves to test the sines of all angles from 0° to 90°, if to θ' be given the different values from 0° to 18°.

For other test formulæ, see exs. 7–11, page 55.

PROB. 2. TO COMPUTE TABLES OF NATURAL TANGENTS, COTANGENTS, SECANTS, AND COSECANTS.

Divide the sines of the angles, in turn, by the cosines; the cosines by the sines; 1 by the cosines; 1 by the sines:

or, replace θ *by* 1′, 2′, 3′ ⋯ *in the formulæ of theor.* 4, *cor.* 1.

PROB. 3. TO COMPUTE TABLES OF LOGARITHMIC FUNCTIONS.

From a table of logarithms of numbers take out the logarithms of the natural sines and cosines :

or, replace θ by $1'$, $2'$, $3'$, $\cdots$ in the formulæ of th. 4, cor. 2.

Subtract the logarithmic cosines from the logarithmic sines ; the logarithmic sines from the logarithmic cosines ; the logarithmic cosines and sines from 0.

THE METHOD OF DIFFERENCES.

A more rapid method is this :

Take out the functions of three, four, or more angles at regular intervals, and find their several "orders of differences"; by the algebraic "method of differences," find the successive terms of the series of logarithms ; interpolate for other angles lying between those of the series, and verify at intervals by direct computation.

For safety, four-place tables must be computed to six places ; five-place tables to seven places, and so on.

When the terms of any order of differences are constant, or differ very little, the rule that follows may be applied to form new terms of the series.

Add the constant difference to the last difference of the next lower order, that sum to the last difference of the next lower order, and so on till a term of the series is reached.

In the example that follows, the numbers below the rules are got by successive addition :

ANGLE.	LOG-SINE.	FIRST DIF.	SECOND DIF.	THIRD DIF.
18°	9.489 9824			
		3 8689		
18° 10′	9.493 8513		−378	
		3 8311		7
18° 20′	9.497 6824		−371	
		3 7940		7
18° 30′	9.501 4764		−364	
		3 7576		7
18° 40′	9.505 2340		−357	
		3 7219		7
18° 50′	9.508 9559		−350	
		3 6869		
19°	9.512 6428			

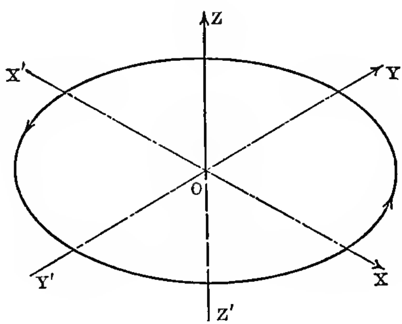
V. SPACE TRIGONOMETRY.

Space Trigonometry treats of the relations of the parts of triedral angles. It is based on the geometry of space, and on the principles established in plane trigonometry.

§ 1. DIRECTED PLANES.

A directed plane was defined on page 25. Such a plane may be generated by a straight line swinging about another straight line that meets it at right angles, in either of two directions.

E.g. let OZ be any straight line, and let OX, perpendicular to OZ, swing about OZ and take in succession the positions OX, OY, OX′, OY′, OX;



then OX generates a plane perpendicular to OZ. [geom.

The fixed line about which the other swings is the *axis of the plane;* and if this axis be so directed that its positive end is in front of the plane, it is a *normal to the plane.*

E.g. in the figure above, OZ is normal to the plane generated by OX, but OZ′ is contra-normal to this plane.

The plane is then also said to be *normal to the line.*

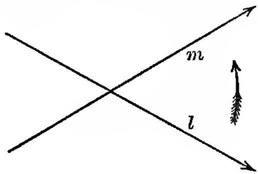
E.g. the plane of the equator is normal to the earth's axis.

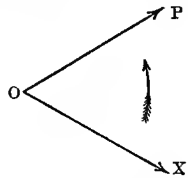
It will be convenient, in this book, to indicate a directed plane by naming two directed lines of the plane in such order that the least rotation about their co-point, from the line first named to the other, generates a positive angle.

E.g. if *l*, *m* be two directed lines that meet, the plane *lm* is a directed plane;

and the plane *ml* is a directed plane that coincides with *lm* in position, but has the contrary direction.

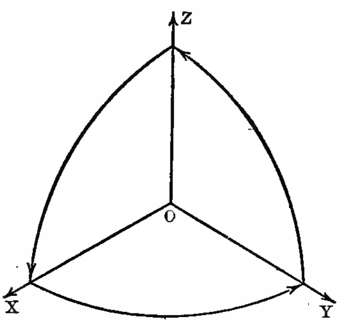
So the plane XOP is a directed plane in which positive rotation is from OX to OP, by the shortest way.

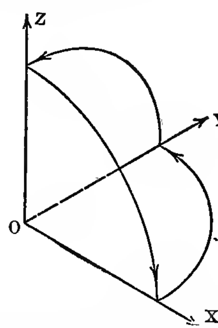




The direction of a plane may also be shown by an arrow.

THEOR. 1. *Three straight lines meeting at a point, and each perpendicular to the other two, may be so directed that each is normal to the plane of the other two taken in order.*





E.g. let OX, OY, OZ be three directed lines such that OX is perpendicular to OY, OZ, and normal to the plane YOZ,

that OY is perpendicular to OZ, OX, and normal to ZOX,

and that OZ is perpendicular to OX, OY, and normal to XOY.

QUESTIONS.

1. If a rod project above a horizontal plane in a direction parallel to the earth's axis, in what direction will its shadow on the plane swing in the northern hemisphere? in the southern hemisphere?

So upon a vertical plane? In what order will the numbers be placed on a horizontal sun-dial? on a vertical sun-dial?

2. To an observer standing behind the transparent dial of a tower clock, what is the direction of rotation of the clock hands? is it the same for all four faces? is the actual direction of rotation the same in two opposite faces?

3. What is a right-hand screw?

4. In turning on the nuts that keep the wheels of a carriage upon the axles, is the motion clockwise or counter-clockwise? is it the same motion on both sides of the carriage?

5. As a carriage is driven forward, how do the wheels turn, to one standing on the right side of the roadway? to one standing on the left side?

6. If when a carriage is driven forward the rotation of the wheels be positive, what is the rotation when the carriage is backing?

7. If a carriage drive past, on which side of the roadway must one stand that the normal to the plane of rotation of the wheels may reach towards him? away from him?

8. How must a line of shafting be directed so that it shall be normal to the pulleys that are fixed upon and revolve with it?

9. If two wheels with parallel axes be so geared that they revolve in opposite directions, what relation have the normals to their planes of rotation?

10. In the figure of theor. 1, how may a point be placed so as to be

in front of all the planes XOY, YOZ, ZOX?

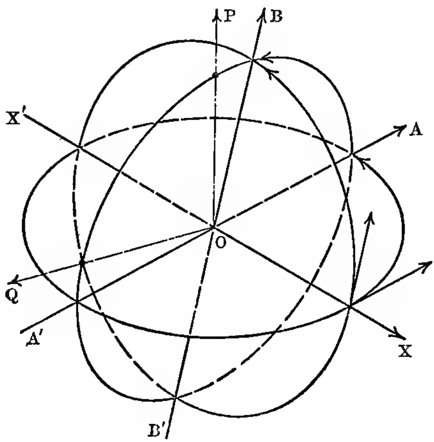
in front of XOY, YOZ, and back of ZOX?

What other positions may a point have?

§ 2. DIEDRAL ANGLES.

If two directed planes meet in a directed line, their *co-line,* and one of them, the *initial plane,* swing about this co-line till it coincides with the other, the *terminal plane,* both in position and direction, the diedral angle so generated is the *angle of the two planes.*

This angle is directed and measured by the plane angle that is generated by a normal to the co-line of the two planes, lying in the initial plane and carried by this plane as it swings about the co-line till it becomes normal to the co-line in the terminal plane. The co-line may be directed at pleasure, but however it is directed the plane of the swinging normal must be taken normal to this line.



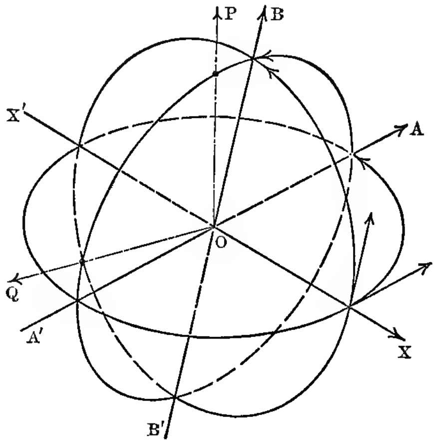
E.g. let the directed planes *a, b* meet in the directed line X′X, and let A′A, B′B be normal, in *a, b,* to X′X at O;

then the diedral angle *ab* is directed and measured by the plane angle AOB, in the plane normal to X′X at O.

So, if the directed co-line be XX′;

then AA′, BB′ are normal in *a, b,* to XX′, at O, the diedral angle *ab* is directed and measured by A′OB′, in the plane normal to XX′ at O.

It is to be noted that the angle A′OB′ as seen from X′ is the opposite of AOB as seen from X, and that the angle B′OA′ is the opposite of BOA; *i.e.* a reversal of the co-line of the two planes reverses their angle.



THEOR. 2. *The angle of two directed planes is equal to the angle of their normals, as seen from the positive end of the directed co-line of the two planes.*

For, in the figure above, draw OP, OQ normal to the planes *a, b*;
then ∵ OP is normal to A′A in the plane AOB, and OQ to B′B,
∴ the angle POQ is equal in magnitude to the angle AOB; [geom.
and ∵ these angles have the same direction in the same plane,
and the plane angle AOB directs and measures the diedral angle *ab*,
∴ the angle of the two planes is equal to the angle of their normals. Q.E.D.

COR. 1. *If the angle ab be a positive right angle, so is the angle* POA; OP *lies in the plane b and coincides with* OB, *and* OQ *lies in the plane a and coincides with* OA′.

NOTE. The student of the geometry and trigonometry of space must train himself to see his figures as figures in space, though shown only by diagrams on a flat surface. For the most part these diagrams are made up of straight lines and curves, and when he looks at the points and lines of his diagrams, he must see the points, lines, and surfaces in space which they represent. It will help him to do this if he will close one eye and, without moving his head, look steadily at his diagram with the other eye: presently it will stand out.

It will help him, also, if he will hold some object, his book for example, or a card, or a wire cage, between the light and the wall: he will learn that the shadows are the pictures, projections, of his space figures on a plane. Among other things, he will see that right angles are rarely projected into right angles, that circles are commonly projected into ellipses and sometimes into straight lines, and that lines of the same length are often unequal; and he will learn to look back from the picture to the figure in space.

E.g. in the diagram on page 104, the horizontal circle seems to be but half as broad as it is long, and the right angles XOY, YOZ are drawn as angles of 60°, while the right angle ZOX is drawn as an angle of 120° and appears to be the sum of the other two.

So, in the figure on page 108, there are three non co-planar straight lines A′A, B′B, X′X, that meet in a point O and determine three planes that meet in the same point. Three circles lie in these planes and have O as their common centre; and these circles determine a sphere whose centre is O.

To make this figure stand out more clearly arcs that lie on the front of the sphere are shown by full lines, while those that are behind either of the other planes are shown by broken lines; and so for the diameters.

The front edge of the horizontal circle is tipped down, while the normal OP is tipped forward and does not show its full length.

§ 3. PROJECTIONS.

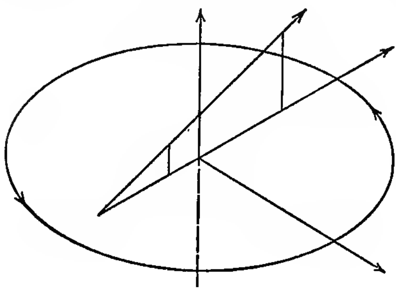
The projection of a point on a line was defined on page 31.

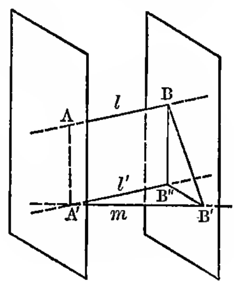
The *projection of a point on a plane* is the foot of the perpendicular from the point to the plane.

The *projection of a directed line on a plane* is the co-line of the given plane and a plane perpendicular to it through the projected line.

The *plane of projection* is that plane on which the projection is made, the perpendicular plane is the *projecting plane,* and the co-line of the two planes is the *line of projection.*

The *angle of a line and a plane* is the angle of the line of projection on the plane, when directed, and the given line.





The *projection of a segment* of a directed line on a plane, or on another directed line, is the segment of the line of projection that reaches from the projection of the initial point of the given segment to that of the terminal point. It is a positive segment if it reach forward, in the direction of the line of projection, a negative segment if it reach backward. The projection of a broken line upon a directed line is the sum of the like projections of the segments that constitute the broken line, and it is equal to the projection of the segment that reaches from the first initial to the last terminal point.

The *angle of two directed lines that do not meet* is that of any two lines parallel to the given lines that meet and reach forward in the same directions as the lines.

THEOR. 3. *If a segment of a directed line be projected on another directed line, the projection is equal to the product of the segment by the cosine of the angle of the two lines.*

(*a*) *The two lines co-planar.* [II, theor. 10.

(*b*) *The two lines not co-planar.*

For, let l, m be two directed lines not co-planar, and let AB be a segment of l, and A′B′ be its projection on m;

through A′ draw l' a line parallel to l and like directed, and through A, B draw planes perpendicular to the line m;

then ∵ these planes are parallel, and A′B″, AB are segments of parallel lines cut off by parallel planes,

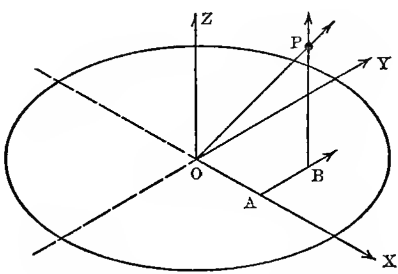
∴ A′B″ = AB; [geom.

and ∵ angle $l'm$ = angle lm, [df. ang. two lines.

and A′B′ = A′B″ cos $l'm$, [II, theor. 10.

∴ A′B′ = AB cos lm. Q.E.D.

COR. *The projection of a broken line upon a directed line is the sum of the products of the segments that constitute the broken line by the cosines of their angles with the line of projection.*



E.g. in the figure above, let OX, OY, OZ be three directed lines, each normal to the plane of the other two;

let P be any point in space, and project P on the plane OXY at B, and B on OX at A;

then the projection of the broken line OABP on OP is OP,

and OP = OA cos XOP + AB cos YOP + BP cos ZOP.

§4. TRIEDRAL ANGLES AND SPHERICAL TRIANGLES.

If three planes meet at a point, they form a *triedral angle.*

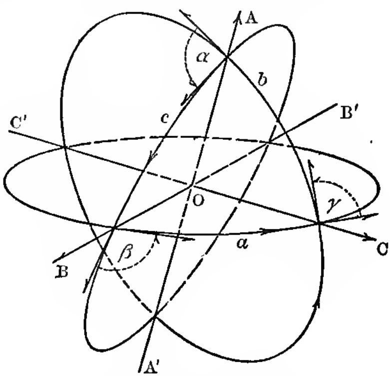
The three face angles and the three diedrals of a triedral are its six *parts.*

If three directed lines be given that meet at a point, they may be taken in such order and their three co-planes may be so directed that all the parts of the triedral shall be positive and less than two right angles; and so, if three directed planes be given, their co-lines may be so taken and directed that all the parts shall be positive and less than two right angles.

E.g. if BOC, COA, AOB be three planes whose directed co-lines are OA, OB, OC,

and if these three planes be so directed that the three face angles BOC, COA, AOB, and the three diedrals COA-AOB, AOB-BOC, BOC-COA are all positive and less than two right angles;

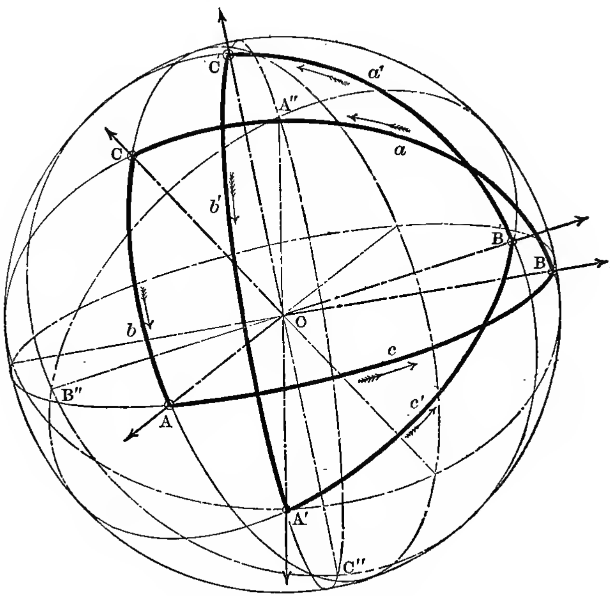
then the triedral O-ABC may be called the *ideal triedral* of the points A, B, C, as to the centre O.



The three directed planes of a triedral BOC, COA, AOB may be named by the three Roman letters *a, b, c,* and so may the three plane angles BOC, COA, AOB; and the three diedrals COA-AOB, AOB-BOC, BOC-COA by the three Greek letters α, β, γ, and so may their three co-lines OA, OB, OC.

POLAR TRIEDRALS.

If through any point normals be drawn to the three faces of a triedral, these normals lie, two and two, in planes perpendicular to the three edges of the triedral [geom.], and if these new planes be so directed that they are normal to the edges of the first triedral, a new triedral is formed so related to the other that the edges of either of them are normal to the faces of the other. Two such triedrals form a pair of *polar triedrals*. The simplest case of such a pair of triedrals is where the six planes all pass through the same point.



THEOR. 4. *In any pair of polar triedrals, the face angles of one of them are equal to the diedrals of the other.*

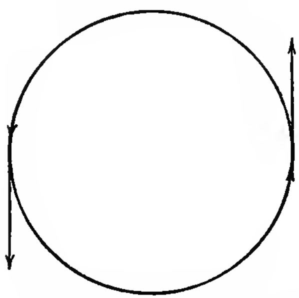
For the angle of a pair of directed planes is equal to that of their normals. [theor. 2.

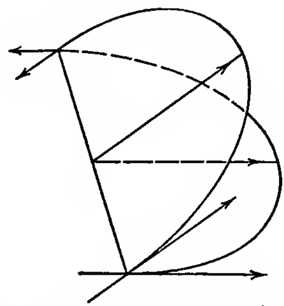
SPHERICAL TRIANGLES.

If any point be taken as the centre of rotation of a directed plane, and a sphere be described about this point as centre, the co-line of the plane and sphere is a circle of rotation of the plane, and so it is a *directed great circle* of the sphere that has the same direction as the plane.

If any diameter of this circle be directed, the tangent at its positive end reaching forward in the direction of the circle is normal to the diameter, and that at its negative end is contra-normal. That diameter of the sphere which is normal to the plane is the *axis* of the great circle, and its ends are the *positive* and *negative poles* of this circle.

E.g. the earth's north and south poles are the positive and negative poles of the plane of the equator.

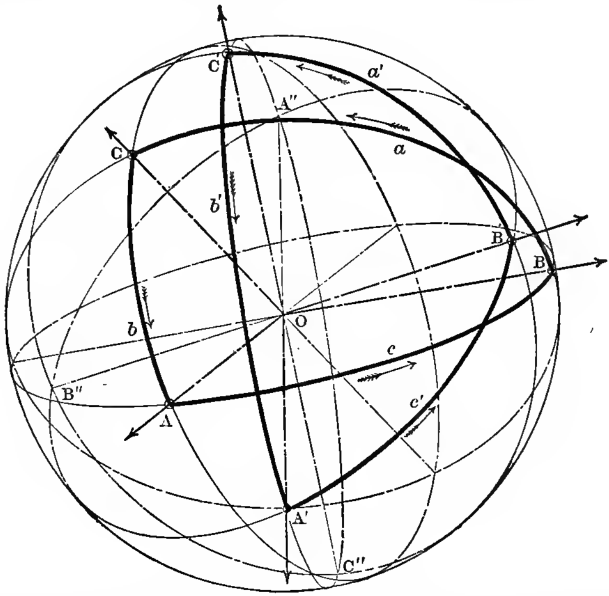




If two directed planes pass through the centre of a sphere, they cut it in two directed great circles; and if their co-diameter be directed, tangents at its positive end that reach forward in the direction of the circles are normal to this diameter, and their angle, in a plane facing the positive end of the diameter, measures the diedral angle of the planes. So the tangents at the negative end of this diameter are contra-normal, and their angle is equal to the other in a plane facing the same way. The angle of the axes of the two circles is equal to that of the two planes. [theor. 2.

E.g. the angle between the plane of the equator and that of the ecliptic, both west-to-east planes, is 23° 27′.

If about the vertex of a triedral angle as centre, a sphere be described, the co-lines of this sphere with the three directed planes are three directed great circles, and together they form a *spherical triangle* whose sides subtend the face angles and whose angles, when viewed from the positive ends of the edges, measure the diedrals of the triedral. The sides meet on the co-diameters of the great circles, *i.e.* on the edges of the triedral, and these points are the vertices of the triangle.



If two polar triedrals have a common vertex, and a sphere be described about this vertex as centre, the six directed circles cut from the six directed planes by this sphere form a *pair of polar spherical triangles,* such that the vertices of the one are the positive poles of the sides of the other, and the sides of the one, measuring the face angles of the triedral, are equal to the angles of the other.

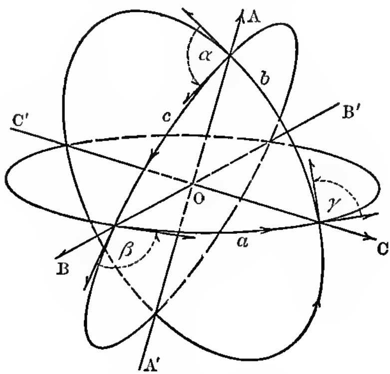
In this figure, the line OA is normal to the plane B'OC', OB to C'OA', OC to A'OB'; OA' to BOC, OB' to COA, OC' to AOB.

THE SIXTY-FOUR TRIEDRALS OF THREE CO-POINTAR LINES.

If A′A, B′B, C′C be three diameters of a sphere that do not lie in the same plane, each of these lines may have either of two directions. It follows that either A or A′ may be taken as the positive end of the diameter A′A, and so for B, B′ and for C, C′, and that there may be eight distinct sets of three points on the surface of the sphere:

A, B, C,	A′, B, C,	A, B′, C,	A, B, C′,
A′, B′, C,	A′, B, C′,	A, B′, C′,	A′, B′, C′,

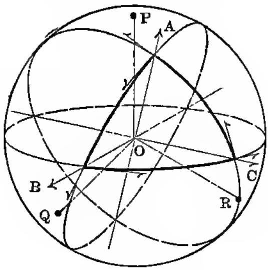
i.e. that these three diameters form eight distinct spherical triangles, and eight distinct triedrals, in the geometric sense.



So, each of the three planes of these three diameters, taken two and two, may have either of two directions, and the triangle of one set of points may have eight distinct forms.

Sixty-four triedrals and sixty-four spherical triangles are thus formed with the same three diameters of a sphere, whose sides are all positive and less than four right angles, and whose angles may be positive or negative.

These triangles are called the *primary triangles*, and other triangles *congruent* with these may be formed by adding multiples of four right angles to either angle, or one or more great circles to either side.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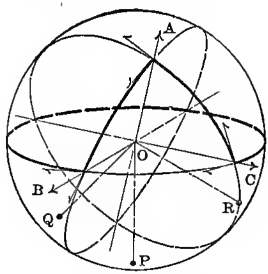
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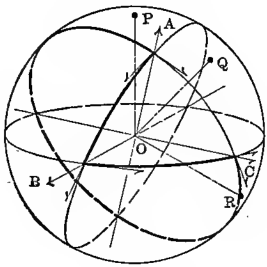
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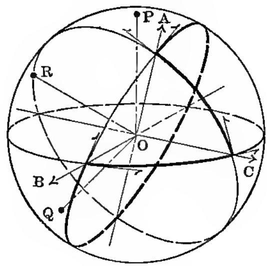
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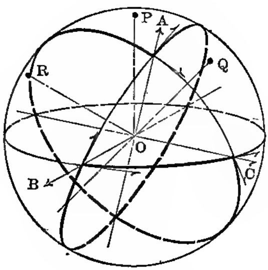
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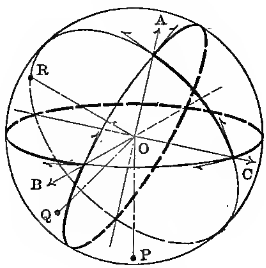
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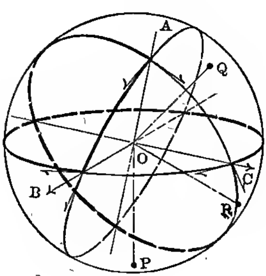
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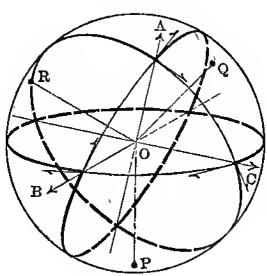
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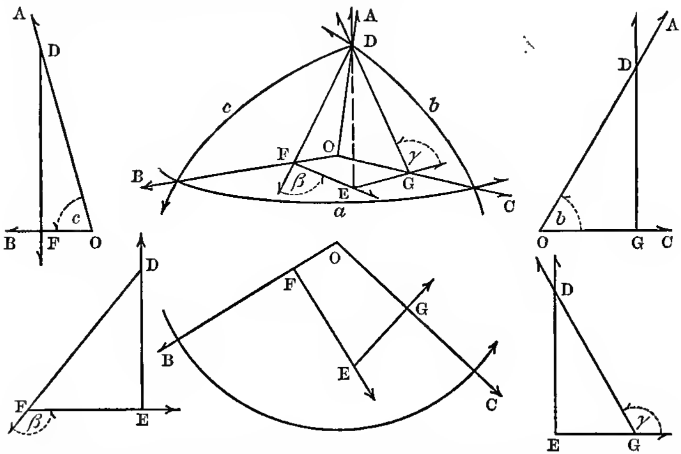
§ 5. GENERAL PROPERTIES OF TRIEDRAL ANGLES.

LEM. 1. *If at any point of an edge of a triedral a normal be drawn to the opposite face, and if through this normal a plane be drawn normal to another edge, the co-lines of this plane with the planes adjacent to the edge are perpendicular to the edge.* [geom.

If these lines be so directed that they are normal to the edge, each in its own plane, the angle of these two normals is equal to the diedral of the two planes. [df. ang. of two planes.

The normal first drawn is normal to that one of the two normals which lies in the opposite face.



E.g. let O-ABC be a triedral angle, through D any point on the edge OA draw ED normal to the opposite face BOC,

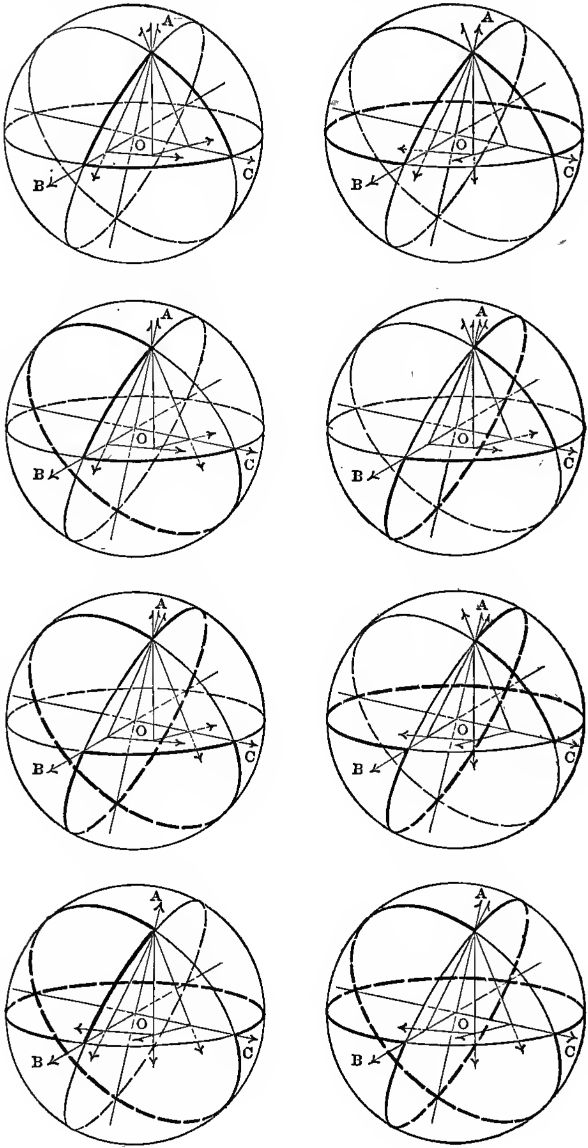
and through ED, draw planes normal to the edges OB, OC, cutting OB in F, and OC in G;

then the lines DF, FE are perpendicular to OB, and EG, GD to OC:

and if DF, FE be directed normal to OB, and EG, GD to OC;

then the plane angle DF-FE is equal to the diedral AOB-BOC, and EG-GD to BOC-COA. [df.

So the line ED, drawn normal to the face BOC, is normal to the line FE in the plane EFD and to the line EG in GED.


A
O
B
C

To make clear the relations of the parts of the figures on page 118, construct a space model as follows:

Use card-board or stiff paper, and with any centre O and any convenient radius draw a circle;

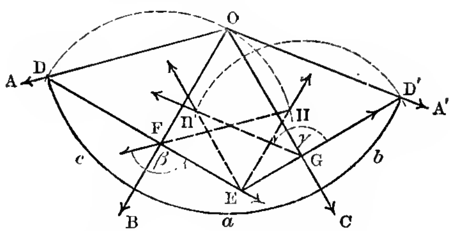
draw the radii OA, OB, OC, OA′, making the angles AOB, BOC, COA′ equal to the given face angles c, a, b;

on OA, OA′ take OD, OD′ equal, and draw DF, GD′ normal to OB, OC at F, G, and meeting each other in E;

cut out the figure, and fold along OB, OC;

bring OA, OA′ together, and join E, D with a thread:

ED is normal to the plane BOC and to the lines FE, EG.



The right triangles FED, EGD are shown in the figure as hinged at FE, EG, and folded down into the plane of the drawing. The point D is shown at H, H′. These triangles turn up when the two faces AOB, COA′ are turned up, and with them they form a solid figure.

Of the six figures on page 118, the upper middle figure is a space figure, the lower middle figure shows the base of this figure in its own plane, and the right triangles, at the right and left, are the right triangles of the space figure, each shown of its true size and in its own plane.

The eight figures on page 119 show the eight spherical triangles of page 117, with the lines ED, DF, FE, EG, GD drawn as in the figure on page 118. The reader will note the directions of these lines, and the consequent directions of the diedrals α, β, γ. The lemma applies to all the figures alike.

§ 6. GRAPHIC SOLUTION OF TRIEDRAL ANGLES.

By a *graphic solution* is meant a geometrical construction of the required figure, such that the parts sought are determined and shown without the use of algebraic formulæ and without computation. Such a solution, often useful of itself and quickly made, serves also as an effective check on the results of numerical computation.

PROB. 1. GIVEN THREE PARTS OF A TRIEDRAL ANGLE, TO CONSTRUCT THE OTHER THREE PARTS:

(*a*) *Given the three face angles, a, b, c:*

Through any point O of the plane of the paper draw *rays* OA, OB, OC, OA′, making the angles AOB, BOC, COA′ equal to the given face angles c, a, b;

with O as centre and any radius, cut OA, OA′ in D, D′;

draw DF, GD′ normal to OB, OC, and meeting each other in E;

through E draw normals to DF, GD′, and cut these normals, on their positive ends, by the circles FD, GD′, *i.e.* by the circles whose centres are F, G, and whose radii are FD, GD′, in H, H′;

join FH, GH′;

then the plane angle HF-FE is equal to the diedral β,

and the plane angle EG-GH′ is equal to the diedral γ.

For, revolve the right triangles FEH, H′EG about FE, EG till EH, EH′ are both normal to the plane a and coincide;

and revolve the right triangles DFO, OGD′ about OB, OC till OA, OA′ coincide in front of the plane a;

then ∵ the right triangles FEH, FED have coincident planes, the same base FE, and equal hypotenuse HF, DF,

∴ the perpendiculars EH, ED are equal.

So, EH′, ED′ are equal, the points H,D, H′,D′ coincide, and the figure of lem. 1 is reproduced;

∴ the plane angles HF-FE, EG-GH′ are equal to the diedrals β, γ.

To construct the diedral α, arrange the face angles in the order a, b, c or b, c, a, and then on as above.

(*b*) *Given the three diedrals, α, β, γ:*

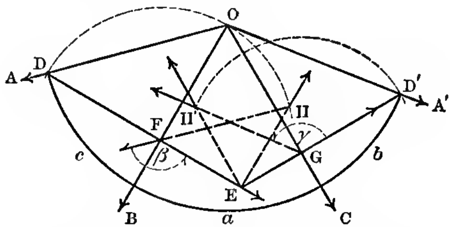
Construct the polar triedral, taking angles equal to α, β, γ for the face angles;

then the three diedrals that are found are equal to the three face angles a, b, c that are sought.

(*c*) *Given a diedral and the two adjacent face angles, β, c, a:*

Through any point O in the plane of the paper, draw rays OA, OB, OC, making the angles AOB, BOC equal to c, a;

with O as centre and any radius, cut OA in D, and draw DF normal to OB at F;



through F draw a line such that the angle of DF with this line is equal to the diedral β, and cut this line on its negative end by the circle FD, in H;

through H draw the normal to DF at E;

draw EG normal to OC, cutting the circle OD at D′, and through D′ draw OA′;

then the angle COA is equal to the face angle b.

The diedrals γ, α may be constructed as in case (a).

(*d*) *Given a face angle and the two adjacent diedrals, b, γ, α:*

Construct the polar triedral, taking angles equal to b, γ, α for a diedral and the two adjacent face angles;

then the face angle and two diedrals that are found are equal to the diedral and two face angles β, c, a that are sought.

(*e*) *Given two face angles and an opposite diedral, b, c, β :*

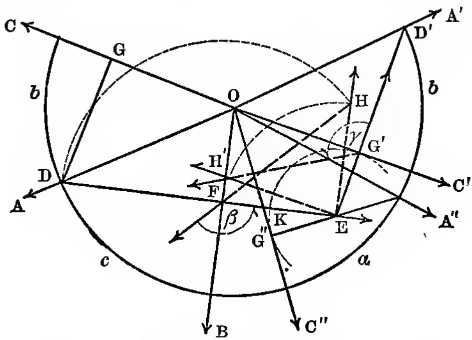
Through any point O in the plane of the paper, draw the rays OC, OA, OB, making the angles COA, AOB equal to the face angles *b*, *c* ;

with O as centre and any radius, cut OA in D ;

through D draw GD normal to OC, and DF normal to OB ;

through F draw a line such that the angle DF makes with this line is the angle *β*,

and with circle FD cut this line on its negative end in H ;



through H draw EH normal to DF, and with H as centre and radius GD cut DE in K ;

and through O draw OC′, OC″ tangent to the circle EK at G′, G″;

then either BOC′ or BOC″ is the face angle *a* ; and the other parts may be constructed as above.

There is no triangle if GD < EH ; one, a right triangle, if GD = EH ; two, if HF > GD > EH ; one, an isosceles triangle, if GD = HF ; one, if GD > HF.

(*f*) *Given two diedrals and an opposite face angle, β, γ, b :*

Construct the polar triangle, taking angles equal to *β*, *γ*, *b* for two sides and a diedral opposite one of them ;

then the face angle and the two diedrals that are found are equal to the diedral and two face angles *α*, *c*, *a*, that are sought.

§ 7. FOUR-PART FORMULÆ.

The reader will note the complete generality of the proof of the LAW OF COSINES and of the LAW OF SINES, no limitation of the sign or magnitude of any part being imposed, and the consequent generality of the formulæ that depend upon these laws.

THE LAW OF COSINES.

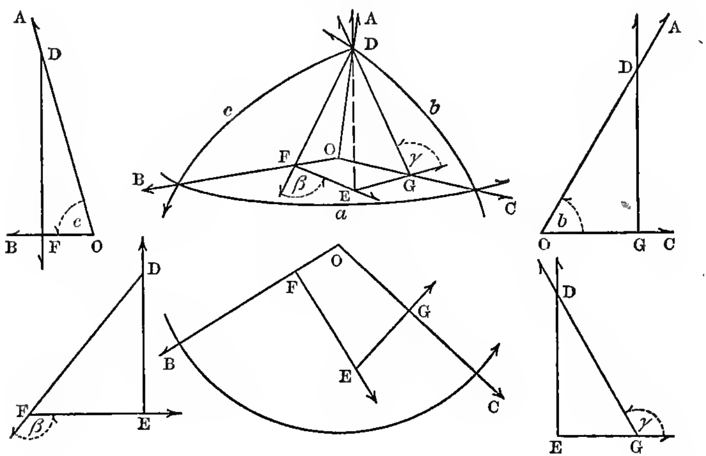
THEOR. 5. *In a triedral angle :*

(*a*) *The cosine of a face angle is equal to the product of the cosines of the other two face angles less the product of their sines by the cosine of the opposite diedral :*

i.e. $\cos a = \cos b \cos c - \sin b \sin c \cos \alpha,$

$\cos b = \cos c \cos a - \sin c \sin a \cos \beta,$

$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$



For, let O-ABC be a triedral angle, through D any point on the edge OA draw ED normal to the opposite face BOC,

and through ED, draw planes normal to the edges OB, OC, cutting OB in F, and OC in G ;

then the lines DF, FE are perpendicular to OB, and EG, GD to OC,

and ∵ the projections on OC of OD and of the broken line OFED are equal, [df.

and proj ED = 0, [ED perp. to OC.

∴ proj OD = proj OF + proj FE,

∴ proj OD/OD = proj OF/OD + proj FE/OD,

∴ proj OD/OD = proj OF/OF · OF/OD
+ proj FE/FE · FE/FD · FD/OD,

∴ cos COA = cos COB · cos BOA + cos OG-FE · cos FE-DF · sin BOA,

i.e. $\cos b = \cos(-a) \cdot \cos(-c) + \cos(-a + \mathrm{R}) \cdot \cos\beta \cdot \sin(-c)$;

∴ $\cos b = \cos c \cos a - \sin c \sin a \cos\beta$. [II, theors. 5, 6.

Q. E. D.

So, ∵ the projections on OB of OD and the broken line OGED are equal,

∴ $\cos c = \cos a \cos b - \sin a \sin b \cos\gamma$. Q. E. D.

So, if D be taken a point on OC, and ED be normal to the face C;
then $\cos a = \cos b \cos c - \sin b \sin c \cos\alpha$. Q. E. D.

The reader may well examine this proof with care: he will see that it is conclusive; but he may ask what suggested the several steps in the tenth and eleventh lines. Only this: it was necessary to eliminate the lines which appear in the equation

proj OD = proj OF + proj FE,

and to bring in the ratios.

Dividing by OD, the first ratio proj OD/OD appears at once as one of the ratios sought. But proj OF/OD is not such a ratio, and the line OF that joins the projection of OF to OD is used as an intermediary line; then the ratio proj OF/OD is written as the product of the two ratios proj OF/OF, OF/OD, which can be interpreted.

So, the ratio proj FE/OD cannot be interpreted, and the two lines FE, FD that join the projection of FE to OD are used as intermediary lines; then the ratio proj FE/OD is written as the product of the three ratios proj FE/FE, FE/FD, FD/OD, which can be interpreted.

(*b*) *The cosine of a diedral angle is equal to the product of the cosines of the other two diedrals less the product of their sines by the cosine of the opposite face angle:*

i.e. $\cos\alpha = \cos\beta \cos\gamma - \sin\beta \sin\gamma \cos a,$

$\cos\beta = \cos\gamma \cos\alpha - \sin\gamma \sin\alpha \cos b,$

$\cos\gamma = \cos\alpha \cos\beta - \sin\alpha \sin\beta \cos c.$

For, let a', b', c', α', β', γ' be the parts of a triedral polar to the given triedral,

then $\because a' = \alpha, \quad b' = \beta, \quad c' = \gamma, \quad \alpha' = a, \quad \beta' = b, \quad \gamma' = c,$ [theor. 4.

and $\cos b' = \cos c' \cos a' - \sin c' \sin a' \cos\beta',$ [above.

$\therefore \cos\beta = \cos\gamma \cos\alpha - \sin\gamma \sin\alpha \cos b.$

So, $\cos\gamma = \cos\alpha \cos\beta - \sin\alpha \sin\beta \cos c,$

$\cos\alpha = \cos\beta \cos\gamma - \sin\beta \sin\gamma \cos a.$ Q.E.D.

COR. 1. $\cos\frac{1}{2}\alpha = \sqrt{[\sin(s-b)\sin(s-c)/\sin b \sin c]},$

[$s \equiv \frac{1}{2}(a+b+c)$.

$\cos\frac{1}{2}a = \sqrt{[\sin(\sigma-\beta)\sin(\sigma-\gamma)/\sin\beta\sin\gamma]},$

[$\sigma \equiv \frac{1}{2}(\alpha+\beta+\gamma)$.

For $\because 2\cos^2\frac{1}{2}\alpha = 1 + \cos\alpha$ [II, theor. 13, cor.

$= 1 + (\cos b \cos c - \cos a)/\sin b \sin c$ [(*a*).

$= (\cos b \cos c + \sin b \sin c - \cos a)/\sin b \sin c$

$= [\cos(b-c) - \cos a]/\sin b \sin c$ [II, theor. 11.

$= -2 \sin\frac{1}{2}(b-c+a) \sin\frac{1}{2}(b-c-a)/\sin b \sin c$ [II, theor. 12.

$= 2 \sin\frac{1}{2}(a-b+c) \sin\frac{1}{2}(a+b-c)/\sin b \sin c$

$= 2 \sin(s-b) \sin(s-c)/\sin b \sin c,$

$\therefore \cos\frac{1}{2}\alpha = \sqrt{[\sin(s-b)\sin(s-c)/\sin b \sin c.]}$ Q.E.D.

So, $\because 2\cos^2\frac{1}{2}a = 1 + \cos a$

$= 1 + (\cos\beta\cos\gamma - \cos\alpha)/\sin\beta\sin\gamma,$

$\therefore \cos\frac{1}{2}a = \sqrt{[\sin(\sigma-\beta)\sin(\sigma-\gamma)/\sin\beta\sin\gamma]}.$

Q.E.D.

COR. 2. $\sin\frac{1}{2}\alpha = \sqrt{[\sin s \sin(s-a)/\sin b \sin c]}$,
$\sin\frac{1}{2}a = \sqrt{[\sin\sigma\sin(\sigma-\alpha)/\sin\beta\sin\gamma]}$.

For $\because 2\sin^2\frac{1}{2}\alpha = 1-\cos\alpha$ [II, theor. 13, cor.

$= 1-(\cos b\cos c-\cos a)/\sin b\sin c$

$= [\cos a-\cos(b+c)]/\sin b\sin c$ [II, theor. 11.

$= -2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a-b-c)/\sin b\sin c$ [II, theor. 12.

$= 2\sin s\sin(s-a)/\sin b\sin c$,

$\therefore \sin\frac{1}{2}\alpha = \sqrt{[\sin s\sin(s-a)/\sin b\sin c]}$. Q.E.D.

So, $\because 2\sin^2\frac{1}{2}a = 1-\cos a$

$= 1-(\cos\beta\cos\gamma-\cos\alpha)/\sin\beta\sin\gamma$,

$\therefore \sin\frac{1}{2}a = \sqrt{[\sin\sigma\sin(\sigma-\alpha)/\sin\beta\sin\gamma]}$.

COR. 3. $\tan\frac{1}{2}\alpha = \sqrt{[\sin s\sin(s-a)/\sin(s-b)\sin(s-c)]}$,
$\tan\frac{1}{2}a = \sqrt{[\sin\sigma\sin(\sigma-\alpha)/\sin(\sigma-\beta)\sin(\sigma-\gamma)]}$.

COR. 4. *If $a, b, c, \alpha, \beta, \gamma$ be all positive and less than two right angles, and* A, B, C *be the interior diedrals, then:*

$\cos a = \cos b\cos c + \sin b\sin c\cos A$,

$\cos A = -\cos B\cos C + \sin B\sin C\cos a$;

$\sin\frac{1}{2}A = \sqrt{[\sin(s-b)\sin(s-c)/\sin b\sin c]}$,

$\sin\frac{1}{2}a = \sqrt{[\sin E\sin(A-E)/\sin B\sin C]}$;

[$E \equiv \frac{1}{2}(A+B+C-2R)$.

$\cos\frac{1}{2}A = \sqrt{[\sin s\sin(s-a)/\sin b\sin c]}$,

$\cos\frac{1}{2}a = \sqrt{[\sin(B-E)\sin(C-E)/\sin B\sin C]}$;

$\tan\frac{1}{2}A = \sqrt{[\sin(s-b)\sin(s-c)/\sin s\sin(s-a)]}$,

$\tan\frac{1}{2}a = \sqrt{[\sin E\sin(A-E)/\sin(B-E)\sin(C-E)]}$.

For $\because$ A, B, C are supplementary to α, β, γ,

$\therefore \cos\alpha = -\cos A$, $\cos\beta = -\cos B$, $\cos\gamma = -\cos C$,

$\sin\alpha = \sin A$, $\sin\beta = \sin B$, $\sin\gamma = \sin C$,

$\sigma = \text{sup } E$, $\sigma-\alpha = A-E$, $\sigma-\beta = B-E$, $\sigma-\gamma = C-E$,

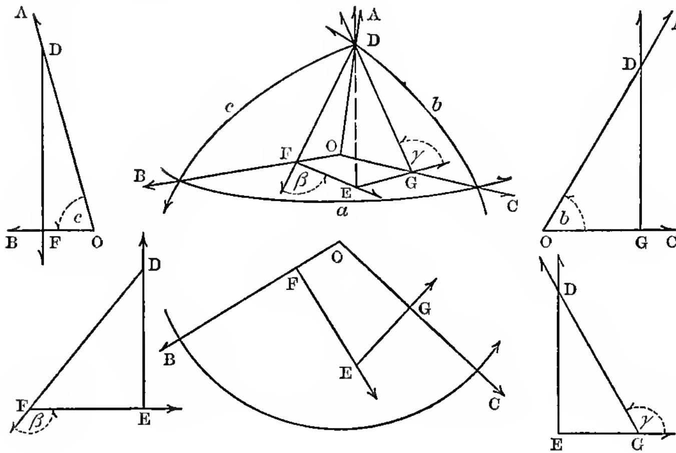
THE LAW OF SINES.

THEOR. 6. *In a triedral angle, the sines of the face angles are proportional to the sines of the opposite diedrals.*

i.e. $\sin a/\sin\alpha = \sin b/\sin\beta = \sin c/\sin\gamma$.

For let O-ABC be a triedral angle, through D any point on the edge OA draw ED normal to the opposite face BOC,

and through ED, draw planes normal to the edges OB, OC, cutting OB in F, and OC in G;



then the lines DF, FE are perpendicular to OB, and EG, GD to OC;

and ∵ the projections of the broken lines OFD, OGD on ED are equal, [df. proj.

and proj OF $= 0$, proj OG $= 0$, [OF, OG perp. to ED.

∴ proj FD $=$ proj GD,

∴ proj FD/OD $=$ proj GD/OD,

∴ proj FD/FD · FD/OD $=$ proj GD/GD · GD/OD,

i.e. sin FE-DF · sin OB-OD $=$ sin EG-GD · sin OC-OD;

∴ $\sin(-\beta)\sin(-c) = \sin\gamma\sin b$,

∴ $\sin\beta\sin c = \sin\gamma\sin b$ and $\sin b/\sin\beta = \sin c/\sin\gamma$.

So, if D be taken a point on OC, and ED be normal to the face c, then $\sin a/\sin \alpha = \sin b/\sin \beta$;

$\therefore \sin a/\sin \alpha = \sin b/\sin \beta = \sin c/\sin \gamma$. Q.E.D.

COR. *If $a, b, c, \alpha, \beta, \gamma$ be all positive and less than two right angles, and* A, B, C *be the interior diedrals, then:*

$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$.

For $\because$ the angles α, A are supplementary, and so are β, B and γ, C,

$\therefore \sin A = \sin \alpha, \quad \sin B = \sin \beta, \quad \sin C = \sin \gamma$. [II, theor. 8.

NOTE 1. If the theorem be regarded as relating to a spherical triangle, it may be written: *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles;* and the law of cosines may be expressed in like form.

QUESTIONS.

If ABC be any spherical triangle, then:

1. $\sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta/\cos \frac{1}{2}\gamma = \sin s/\sin c$,
2. $\sin \frac{1}{2}a \sin \frac{1}{2}b/\cos \frac{1}{2}c = \sin \sigma/\sin \gamma$,
3. $\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta/\cos \frac{1}{2}\gamma = \sin (s-c)/\sin c$,
4. $\cos \frac{1}{2}a \cos \frac{1}{2}b/\cos \frac{1}{2}c = \sin (\sigma-\gamma)/\sin \gamma$,
5. $\sin \frac{1}{2}\alpha \cos \frac{1}{2}\beta/\sin \frac{1}{2}\gamma = \sin (s-a)/\sin c$,
6. $\sin \frac{1}{2}a \cos \frac{1}{2}b/\sin \frac{1}{2}c = \cos (\sigma-a)/\sin \gamma$,
7. $\cot \frac{1}{2}\alpha/\cot \frac{1}{2}\gamma = \sin (s-c)/\sin (s-a)$,
8. $\cot \frac{1}{2}a/\cot \frac{1}{2}c = \sin (\sigma-\gamma)/\sin (\sigma-\alpha)$,
9. $\cot \frac{1}{2}\beta \cot \frac{1}{2}\gamma = -\sin (s-a)/\sin s$,
10. $\cot \frac{1}{2}b \cot \frac{1}{2}c = -\sin (\sigma-\alpha)/\sin \sigma$,
11. $\sin (s-a) \cot \frac{1}{2}\alpha = \sin (s-b) \cot \frac{1}{2}\beta = \sin (s-c) \cot \frac{1}{2}\gamma$,
12. $\sin (\sigma-\alpha) \cot \frac{1}{2}a = \sin (\sigma-\beta) \cot \frac{1}{2}b = \sin (\sigma-\gamma) \cot \frac{1}{2}c$.

NOTE 2. The law of sines may be proved by aid of the law of cosines as follows:

For $\because \cos a = \cos b \cos c - \sin b \sin c \cos \alpha,$

$\therefore \cos \alpha = (\cos b \cos c - \cos a)/\sin b \sin c,$

$\therefore \sin^2\alpha, = 1 - \cos^2\alpha, = 1 - (\cos b \cos c - \cos a)^2/\sin^2 b \sin^2 c,$

$$\begin{aligned}\therefore \sin^2\alpha/\sin^2 a &= [\sin^2 b \sin^2 c - (\cos b \cos c - \cos a)^2]/\sin^2 a \sin^2 b \sin^2 c \\ &= [1 - \cos^2 a - \cos^2 b - \cos^2 c \\ &\qquad + 2 \cos a \cos b \cos c]/\sin^2 a \sin^2 b \sin^2 c;\end{aligned}$$

and $\because$ this value is symmetric as to $a, b, c,$

$\therefore \sin^2\beta/\sin^2 b, \quad \sin^2\gamma/\sin^2 c$ have the same value,

$\therefore \sin^2\alpha/\sin^2 a = \sin^2\beta/\sin^2 b = \sin^2\gamma/\sin^2 c.$ Q.E.D.

Or as follows:

$$\begin{aligned}\because \sin^2\alpha &= [\sin^2 b \sin^2 c - (\cos b \cos c - \cos a)^2]/\sin^2 b \sin^2 c, \\ &= [\sin b \sin c - \cos b \cos c + \cos a] \\ &\qquad \cdot [\sin b \sin c + \cos b \cos c - \cos a]/\sin^2 b \sin^2 c \\ &= [\cos a - \cos (b+c)] \cdot [\cos (b-c) - \cos a]/\sin^2 b \sin^2 c \\ &= 4 \sin \tfrac{1}{2}(a+b+c) \cdot \sin \tfrac{1}{2}(b+c-a) \cdot \sin \tfrac{1}{2}(a-b+c) \\ &\qquad \cdot \sin \tfrac{1}{2}(a+b-c)/\sin^2 b \sin^2 c,\end{aligned}$$

$$\therefore \sin^2\alpha/\sin^2 a = 4 \sin s \cdot \sin (s-a) \cdot \sin (s-b) \cdot \sin (s-c) /\sin^2 a \sin^2 b \sin^2 c,$$

and this value is symmetric as to $a, b, c,$

$\therefore \sin^2\alpha/\sin^2 a = \sin^2\beta/\sin^2 b = \sin^2\gamma/\sin^2 c.$ Q.E.D.

NOTE 3. By v. Staudt the expression

$$\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}$$

is called the *sine of the spherical triangle*, and the expression

$$\sqrt{(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2 \cos \alpha \cos \beta \cos \gamma)}$$

the *sine of the polar triangle*. Casey has called them the *first staudtian* and *second staudtian*. The first staudtian is equal to either of the products,

$$\sin b \sin c \sin \alpha, \quad \sin c \sin a \sin \beta, \quad \sin a \sin b \sin \gamma;$$

and the second staudtian to either of the products,

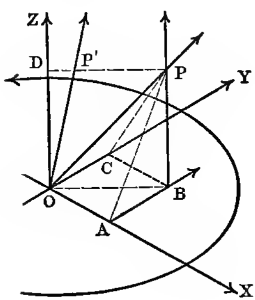
$$\sin \beta \sin \gamma \sin a, \quad \sin \gamma \sin \alpha \sin b, \quad \sin \alpha \sin \beta \sin c.$$

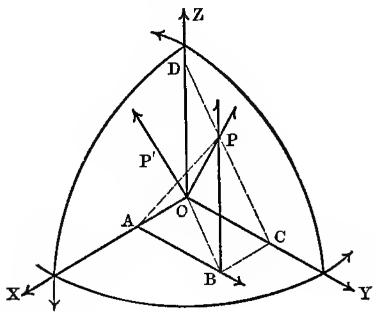
§ 8. ANGLES BETWEEN LINES IN SPACE, AND BETWEEN PLANES.

Let OX, OY, OZ be three lines through a point O, so directed that each is normal to the plane of the other two taken in order,

i.e. so that OX is normal to the plane YOZ, OY to ZOX, OZ to XOY;

then also the angles YOZ, ZOX, XOY are positive right angles as seen from X, Y, Z.





Let OP be any other line through O;

then OP is completely determined by the angles XOP, YOP, ZOP.

Let l, m, $n \equiv$ cos XOP, cos YOP, cos ZOP;

then l, m, n are the *direction cosines* of OP, and determine it.

So, l, m, n and a point determine a plane through the point, normal to OP, and l, m, n are called the *direction cosines of the plane.*

The direction cosines of a line not through O are the direction cosines of a line through O parallel to the given line.

THEOR. 7. *If l, m, n be the direction cosines of a line in space, then* $l^2+m^2+n^2=1$.

For draw OP parallel to the given line, through P draw a line parallel to OZ, meeting the plane XOY in B;

through B draw a line parallel to OY, meeting OX in A;

then ∵ OA, AB, BP are the non-parallel edges of a rectangular parallelopiped and OP is its diagonal,

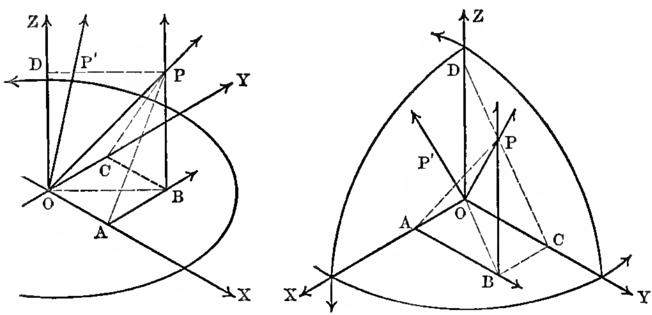
∴ $OA^2 + OB^2 + BP^2 = OP^2$ and $OA^2/OP^2 + OB^2/OP^2 + BP^2/OP^2 = 1$;

i.e. $l^2 + m^2 + n^2 = 1$. Q.E.D. [df. dir. cos.

THEOR. 8. *If l, m, n, l', m', n' be the direction cosines of two directed lines in space, and α their angle, then:*

$$\cos \alpha = ll' + mm' + nn'.$$

For through O draw OP, OP′ parallel to the two given lines, and draw BP normal to the plane XOY at B, and AB normal to OX at A;



project OP and the broken line OABP on OP′;

then ∵ proj OP = proj OA + proj AB + proj BP,

∴ proj OP/OP = proj OA/OP + proj AB/OP + proj BP/OP

= proj OA/OA · OA/OP + proj AB/AB · AB/OP + proj BP/BP · BP/OP;

and ∵ OA = proj OP on OX,

AB = proj OP on OY,

BP = proj OP on OZ,

∴ cos α = cos XOP′ · cos XOP + cos YOP′ · cos YOP + cos ZOP′ · cos ZOP,

i.e. $\cos \alpha = ll' + mm' + nn'$. Q.E.D.

COR. *If α be a right angle, then $ll' + mm' + nn' = 0$.*

THEOR. 9. *If* $l, m, n,\quad l', m', n'$ *be the direction cosines of two directed lines that meet in space,* α *their angle, and* λ, μ, ν *the direction cosines of their plane, then:*

$$\lambda = (mn' - m'n)/\sin\alpha,$$
$$\mu = (nl' - n'l)/\sin\alpha,$$
$$\nu = (lm' - l'm)/\sin\alpha.$$

For $\because$ the normal to the plane is perpendicular to every line of the plane, and so to the two given lines,

$\therefore l\lambda + m\mu + n\nu = 0,\quad l'\lambda + m'\mu + n'\nu = 0;$ [theor. 8, cor.

and $\because \lambda^2 + \mu^2 + \nu^2 = 1,$ [theor. 7.

$\therefore \lambda = (mn' - m'n)$ [solve for λ.

$/\sqrt{[(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2]};$

and $\because l^2 + m^2 + n^2 = 1,\quad l'^2 + m'^2 + n'^2 = 1,$ [theor. 7.

$\therefore \lambda = (mn' - m'n)/\sqrt{[1 - (ll' + mm' + nn')^2]},$

$= (mn' - m'n)/\sin\alpha$; and so for μ, ν. Q.E.D.

NOTE. A new proof of the LAW OF COSINES may be made from the principles established in theors. 7, 8, 9.

Let α, β, γ be three directed lines in space that meet and form a triedral angle, whose face angles are a, b, c and whose opposite diedrals are α, β, γ, as shown in § 4;

let $l, m, n,\quad l', m', n',\quad l'', m'', n''$ be the direction cosines of the lines α, β, γ,

and $\lambda, \mu, \nu,\quad \lambda', \mu', \nu',\quad \lambda'', \mu'', \nu''$ be the direction cosines of the planes $\beta\gamma,\ \gamma\alpha,\ \alpha\beta$, *i.e.* of the planes a, b, c;

then $\because \lambda = (m'n'' - m''n')/\sin a,\ \lambda' = (m''n - mn'')/\sin b,$ [th. 9.

$\mu = (n'l'' - n''l')/\sin a,\quad \mu' = (n''l - nl'')/\sin b,$

$\nu = (l'm'' - l''m')/\sin a,\quad \nu' = (l''m - lm'')/\sin b;$

and $\because \cos\gamma = \lambda\lambda' + \mu\mu' + \nu\nu',$ [theor. 8.

$\therefore \cos\gamma = [(m'n'' - m''n')(m''n - mn'') + (n'l'' - n''l')$
$(n''l - nl'') + (l'm'' - l''m')(l''m - lm'')]/\sin a \sin b$

$= [(l'l'' + m'm'' + n'n'')(l''l + m''m + n''n)$
$- (ll' + mm' + nn')(l''^2 + m''^2 + n''^2)]/\sin a \sin b$

$= [\cos a \cos b - \cos c]/\sin a \sin b,$ [theors. 7, 8.

$\therefore \cos c = \cos a \cos b - \sin a \sin b \cos\gamma.$

So, $\because \cos\alpha = \lambda'\lambda'' + \mu'\mu'' + \nu'\nu''$

$= (\cos b \cos c - \cos a)/\sin b \sin c,$

$\therefore \cos a = \cos b \cos c - \sin b \sin c \cos\alpha.$

So, $\because \cos\beta = \lambda''\lambda + \mu''\mu + \nu''\nu$

$= (\cos c \cos a - \cos b)/\sin c \sin a,$

$\therefore \cos b = \cos c \cos a - \sin c \sin a \cos\beta.$

So, $\because \cos c = ll' + mm' + nn'$

$= (\cos\alpha + \cos\beta - \cos\gamma)/\sin\alpha \sin\beta,$

$\therefore \cos\gamma = \cos\alpha \cos\beta - \sin\alpha \sin\beta \cos c;$

and so for $\cos\alpha$, $\cos\beta$.

§9. FIVE-PART FORMULÆ.

THEOR. 10. *In a triedral angle whose parts are $a, b, c, \alpha, \beta, \gamma$,*

$\sin b \cos\gamma + \cos c \sin a + \sin c \cos a \cos\beta = 0,$

$\sin c \cos\beta + \cos b \sin a + \sin b \cos a \cos\gamma = 0;$

$\sin c \cos\alpha + \cos a \sin b + \sin a \cos b \cos\gamma = 0,$

$\sin a \cos\gamma + \cos c \sin b + \sin c \cos b \cos\alpha = 0;$

$\sin a \cos\beta + \cos b \sin c + \sin b \cos c \cos\alpha = 0,$

$\sin b \cos\alpha + \cos a \sin c + \sin a \cos c \cos\beta = 0;$

For, project the broken line GOFE OR EG;

then $\because$ proj GO $=0$, [GO perp. to GE.

$\therefore$ proj OF + proj FE = GE,

$\therefore$ proj OF/OD + proj FE/OD = GE/OD,

$\therefore$ proj OF/OF · OF/OD + proj FE/FE · FE/FD · FD/OD

= GE/GD · GD/OD,

$\therefore$ cos EG-OF · cos BOA + cos EG-FE · cos FE-FD · sin BOA

= cos EG-GD · sin COA,

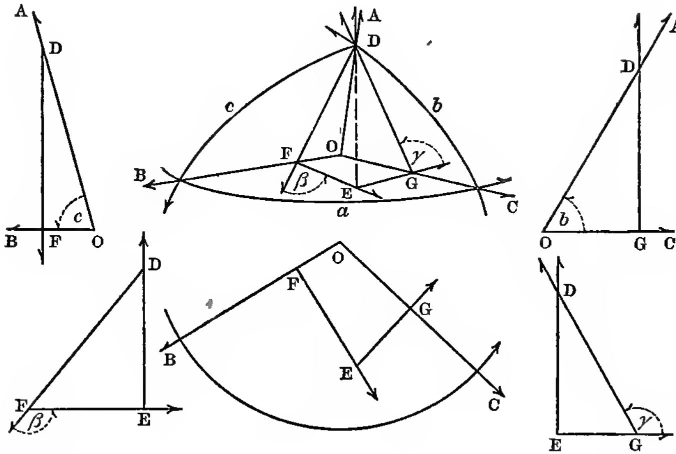
i.e. $\cos(-R-a)\cdot\cos(-c) + \cos(-a)\cdot\cos(-\beta)\cdot\sin(-c)$

$= \cos\gamma\cdot\sin b;$

$\because -\sin a \cos c - \cos a \cos\beta \sin c = \cos\gamma \sin b,$

$\therefore \sin b \cos\gamma + \cos c \sin a + \sin c \cos a \cos\beta = 0.$ Q.E.D.

So, project the broken line FOGE on FE;



then ∵ proj FO = 0, [FO perp. to FE.

$\therefore$ proj OG + proj GE = FE,

$\therefore$ proj OG/OD + proj GE/OD = FE/OD,

$\therefore$ proj OG/OG · OG/OD + proj GE/GE · GE/GD · GD/OD
$= $ FE/FD · FD/OD,

i.e. $\cos(\mathrm{R}-a)\cdot\cos$ COA + cos FE-GE · cos GE-GD · sin COA
$=$ cos FE-FD · sin BOA;

$\therefore \sin a\cdot\cos b+\cos(2\mathrm{R}+a)\cdot\cos(-\sup\gamma)\cdot\sin b$
$=\cos(-\beta)\cdot\sin(-c)$,

$\therefore \sin a\,\cos b+\cos a\,\cos\gamma\,\sin b=-\cos\beta\,\sin c$,

$\therefore \sin c\,\cos\beta+\cos\beta\,\sin a+\sin b\,\cos a\,\cos\gamma=0.$ Q.E.D.

So, with normals drawn to the planes b, c, in turn, the other four formulæ may be proved directly;

or they may be inferred by symmetry, from the two formulæ just proved, *i.e.* the third and fifth from the first, and the fourth and sixth from the second.

COR. $\sin\beta\cos\gamma+\cos c\sin\alpha+\sin\gamma\cos a\cos\beta=0,$

$\sin\gamma\cos\beta+\cos b\sin\alpha+\sin\beta\cos a\cos\gamma=0;$

$\sin\gamma\cos\alpha+\cos a\sin\beta+\sin\alpha\cos b\cos\gamma=0,$

$\sin\alpha\cos\gamma+\cos c\sin\beta+\sin\gamma\cos b\cos\alpha=0;$

$\sin\alpha\cos\beta+\cos b\sin\gamma+\sin\beta\cos c\cos\alpha=0,$

$\sin\beta\cos\alpha+\cos a\sin\gamma+\sin\alpha\cos c\cos\beta=0.$

For $\because \sin a/\sin\alpha=\sin b/\sin\beta=\sin c/\sin\gamma,$ [law of sines.

$\therefore \sin a,\ \sin b,\ \sin c$ may be replaced by $\sin\alpha,\ \sin\beta,\ \sin\gamma$ in the formulæ of the theorem, and those of the corollary result directly.

NOTE 1. The formulæ of the theorem and those of the corollary may be paired in such manner that, if one of them be taken as applying to a triangle, the other is seen to be true for the polar triangle.

E.g. the fourth formula of the corollary may be paired with the first formula of the theorem.

Such a pair of formulæ may be called a pair of *polar formulæ*.

NOTE 2. The law of cosines may be proved by aid of the formulæ given above, and without the polar triedral.

E.g. Multiply the first formula of the corollary by $\cos\beta$ and subtract the product from the last formula; then $\cos c$ is eliminated;

and $\because \sin\beta\cos\alpha-\sin\beta\cos\beta\cos\gamma+\cos a\sin\gamma(1-\cos^2\beta)=0,$

$\therefore \cos\alpha=\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos a.$ Q.E.D.

So, conversely, the formulæ of theor. 10 may be found from the law of cosines by retracing these steps.

QUESTIONS.

Which formula of the corollary may be paired with the second formula of the theorem? with the third? with the fourth? with the fifth? with the sixth?

Rewrite the formulæ of the corollary so as to show their correlation with those of the theorem, letter for letter and term for term.

NAPIER'S ANALOGIES.

THEOR. 11. *In a triedral angle whose parts are* $a, b, c, \alpha, \beta, \gamma,$

$$\tan\tfrac{1}{2}(\beta+\gamma)/\tan\tfrac{1}{2}\alpha = -\cos\tfrac{1}{2}(b-c)/\cos\tfrac{1}{2}(b+c),$$
$$\tan\tfrac{1}{2}(\beta-\gamma)/\tan\tfrac{1}{2}\alpha = -\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(b+c),$$
$$\tan\tfrac{1}{2}(b+c)/\tan\tfrac{1}{2}a = -\cos\tfrac{1}{2}(\beta-\gamma)/\cos\tfrac{1}{2}(\beta+\gamma),$$
$$\tan\tfrac{1}{2}(b-c)/\tan\tfrac{1}{2}a = -\sin\tfrac{1}{2}(\beta-\gamma)/\sin\tfrac{1}{2}(\beta+\gamma).$$

For, add the fourth and fifth equations of theor. 10,

then $\sin a\,(\cos\beta+\cos\gamma)+(1+\cos\alpha)\,\sin(b+c)=0$;

and $\because \sin a/\sin\alpha=\sin b/\sin\beta=\sin c/\sin\gamma$

$$=(\sin b+\sin c)/(\sin\beta+\sin\gamma),$$

$$\therefore \sin a=(\sin b+\sin c)\,\sin\alpha/(\sin\beta+\sin\gamma),$$

$$\therefore (1+\cos\alpha)\,\sin(b+c)$$
$$=-(\sin b+\sin c)\sin\alpha\,(\cos\beta+\cos\gamma)/(\sin\beta+\sin\gamma),$$

$$\therefore (\sin\beta+\sin\gamma)/(\cos\beta+\cos\gamma)\cdot(1+\cos\alpha)/\sin\alpha$$
$$=-(\sin b+\sin c)/\sin(b+c);$$

$$\therefore 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma)$$
$$/2\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma)\cdot\cot\tfrac{1}{2}\alpha$$
$$=-2\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}(b-c)/2\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}(b+c),$$

[II, theor. 12.

$$\therefore \tan\tfrac{1}{2}(\beta+\gamma)/\tan\tfrac{1}{2}\alpha=-\cos\tfrac{1}{2}(b-c)/\cos\tfrac{1}{2}(b+c).$$ Q.E.D.

So, $\because \sin a/\sin\alpha=(\sin b-\sin c)/(\sin\beta-\sin\gamma)$,

$$\therefore (1+\cos\alpha)\,\sin(b+c)$$
$$=-(\sin b-\sin c)\sin\alpha\,(\cos\beta+\cos\gamma)/(\sin\beta-\sin\gamma),$$

and $\tan\tfrac{1}{2}(\beta-\gamma)/\tan\tfrac{1}{2}\alpha=-\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(b+c)$.

So, add the first and second equations of theor. 10, cor.,

then $\sin\alpha\,(\cos b+\cos c)+(1+\cos a)\,\sin(\beta+\gamma)=0$;

and $\because \sin\alpha=\sin a\,(\sin\beta+\sin\gamma)/(\sin b+\sin c)$,

$$\therefore (1+\cos a)\,\sin(\beta+\gamma)$$
$$=-(\sin\beta+\sin\gamma)\sin a\,(\cos b+\cos c)/(\sin b+\sin c),$$

and $\tan\tfrac{1}{2}(b+c)/\tan\tfrac{1}{2}a=-\cos\tfrac{1}{2}(\beta-\gamma)/\cos\tfrac{1}{2}(\beta+\gamma)$.

So, $\because \sin\alpha = \sin a\,(\sin\beta - \sin\gamma)/(\sin b - \sin c)$,

$$\therefore (1+\cos a)\sin(\beta+\gamma)$$
$$= -(\sin\beta - \sin\gamma)/\sin a\,(\cos\beta + \cos\gamma)/(\sin b - \sin c),$$

and $\quad \tan\frac{1}{2}(b-c)/\tan\frac{1}{2}a = -\sin\frac{1}{2}(\beta-\gamma)/\sin\frac{1}{2}(\beta+\gamma)$.

COR. *If $a, b, c, \alpha, \beta, \gamma$ be all positive and less than two right angles, and* A, B, C *be the interior diedrals, then :*

$$\tan\tfrac{1}{2}(B+C)/\cot\tfrac{1}{2}A = \cos\tfrac{1}{2}(b-c)/\cos\tfrac{1}{2}(b+c),$$
$$\tan\tfrac{1}{2}(B-C)/\cot\tfrac{1}{2}A = \sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(b+c),$$
$$\tan\tfrac{1}{2}(b+c)/\tan\tfrac{1}{2}a = \cos\tfrac{1}{2}(B-C)/\cos\tfrac{1}{2}(B+C),$$
$$\tan\tfrac{1}{2}(b-c)/\tan\tfrac{1}{2}a = \sin\tfrac{1}{2}(B-C)/\sin\tfrac{1}{2}(B+C).$$

NOTE. Another proof of Napier's analogies is given below : it does not use the formula of theor. 10, and it has the single defect that it employs radicals, and so is not free from ambiguous signs.

$\tan\frac{1}{2}(\beta+\gamma)/\tan\frac{1}{2}\alpha$ [II, theor. 11, cor. 1.

$$= (\tan\tfrac{1}{2}\beta + \tan\tfrac{1}{2}\gamma)/\tan\tfrac{1}{2}\alpha\,(1 - \tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma)$$

$$= \frac{\sqrt{\dfrac{\sin s\sin(s-b)}{\sin(s-c)\sin(s-a)}} + \sqrt{\dfrac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}}{\sqrt{\dfrac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}\cdot\left[1 - \dfrac{\sin s}{\sin(s-a)}\right]}$$

[theor. 5, cor. 3.

Strike out the common factor $\sqrt{\sin s}$, and multiply both numerator and denominator by $\sqrt{\sin(s-a)\sin(s-b)\sin(s-c)}$; then $\quad \tan\frac{1}{2}(\beta+\gamma)/\tan\frac{1}{2}\alpha$

$$= [\sin(s-b) + \sin(s-c)]/[\sin(s-a) - \sin s]$$
$$= \sin\tfrac{1}{2}a\cos\tfrac{1}{2}(b-c)/-\sin\tfrac{1}{2}a\cos\tfrac{1}{2}(b+c) \quad \text{[II, th. 12.}$$
$$= -\cos\tfrac{1}{2}(b-c)/\cos\tfrac{1}{2}(b+c)\,; \quad \text{Q.E.D.}$$

and so for the rest.

§ 10. SIX-PART FORMULÆ.

DELAMBRE'S FORMULÆ.

THEOR. 12. *In a triedral angle whose parts are $a, b, c, \alpha, \beta, \gamma$,*

$$\sin\tfrac{1}{2}a/\sin\tfrac{1}{2}\alpha = \mp\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\beta-\gamma),$$
$$\sin\tfrac{1}{2}a/\cos\tfrac{1}{2}\alpha = \pm\sin\tfrac{1}{2}(b+c)/\cos\tfrac{1}{2}(\beta-\gamma),$$
$$\cos\tfrac{1}{2}a/\sin\tfrac{1}{2}\alpha = \pm\cos\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\beta+\gamma),$$
$$\cos\tfrac{1}{2}a/\cos\tfrac{1}{2}\alpha = \mp\cos\tfrac{1}{2}(b+c)/\cos\tfrac{1}{2}(\beta+\gamma),$$

with like formulæ if a, α be replaced by b, β or by c, γ:

For $\because \sin^2 a/\sin^2\alpha \equiv (1+\cos a)(1-\cos a)/(1+\cos\alpha)(1-\cos\alpha)$,

and $\sin^2 a/\sin^2\alpha = \sin b \sin c/\sin\beta\sin\gamma$, [law of sines.

$$\therefore (1-\cos a)/(1-\cos\alpha)$$
$$= (1+\cos\alpha)\sin b\sin c/(1+\cos a)\sin\beta\sin\gamma$$
$$= [(1-\cos a)-(1+\cos\alpha)\sin b\sin c]$$
$$/[(1-\cos\alpha)-(1+\cos a)\sin\beta\sin\gamma] \quad \text{[prop.}$$
$$= [1-(\cos a+\sin b\sin c\cos\alpha)-\sin b\sin c]$$
$$/[1-(\cos\alpha+\sin\beta\sin\gamma\cos a)-\sin\beta\sin\gamma]$$
$$= [1-\cos b\cos c-\sin b\sin c]$$
$$/[1-\cos\beta\cos\gamma-\sin\beta\sin\gamma] \quad \text{[law of cos.}$$
$$= [1-\cos(b-c)]/[1-\cos(\beta-\gamma)], \quad \text{[add. theor.}$$
$$\therefore 2\sin^2\tfrac{1}{2}a/2\sin^2\tfrac{1}{2}\alpha = 2\sin^2\tfrac{1}{2}(b-c)/2\sin^2\tfrac{1}{2}(\beta-\gamma),$$
$$\therefore \sin\tfrac{1}{2}a/\sin\tfrac{1}{2}\alpha = \mp\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\beta-\gamma).$$

So, $(1-\cos a)/(1+\cos\alpha)$

$$= (1-\cos\alpha)\sin b\sin c/(1+\cos a)\sin\beta\sin\gamma,$$

and $\sin\tfrac{1}{2}a/\cos\tfrac{1}{2}\alpha = \pm\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}(\beta-\gamma)$.

So, $(1+\cos a)/(1-\cos\alpha)$

$$= (1+\cos\alpha)\sin b\sin c/(1-\cos a)\sin\beta\sin\gamma,$$

and $\cos\tfrac{1}{2}a/\sin\tfrac{1}{2}\alpha = \pm\cos\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\beta+\gamma)$.

So, $(1+\cos a)/(1+\cos\alpha)$

$$= (1-\cos\alpha)\sin b\sin c/(1-\cos a)\sin\beta\sin\gamma,$$

and $\cos\tfrac{1}{2}a/\cos\tfrac{1}{2}\alpha = \mp\cos\tfrac{1}{2}(b+c)/\cos\tfrac{1}{2}(\beta+\gamma)$.

NOTE 1. For any triangle the second members of these equations must all have their upper signs or all their lower signs, since Napier's analogies may be got from Delambre's formulæ by division, and must accord with them:

first Nap. anal. from third Del. form. by fourth Del. form.,
second Nap. anal. from first Del. form. by second Del. form.,
third Nap. anal. from second Del. form. by fourth Del. form.,
fourth Nap. anal. from first Del. form. by third Del. form.

COR. *If $a, b, c, \alpha, \beta, \gamma$ be all positive and less than two right angles, and* A, B, C *be the interior diedrals, then:*

$$\sin\tfrac{1}{2}a/\cos\tfrac{1}{2}\text{A} = +\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\text{B}-\text{C}),$$
$$\sin\tfrac{1}{2}a/\sin\tfrac{1}{2}\text{A} = +\sin\tfrac{1}{2}(b+c)/\cos\tfrac{1}{2}(\text{B}-\text{C}),$$
$$\cos\tfrac{1}{2}a/\cos\tfrac{1}{2}\text{A} = +\cos\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\text{B}+\text{C}),$$
$$\cos\tfrac{1}{2}a/\sin\tfrac{1}{2}\text{A} = +\cos\tfrac{1}{2}(b+c)/\cos\tfrac{1}{2}(\text{B}+\text{C}).$$

For $\because$ A, B, C are the supplements of α, β, γ,

$\therefore \sin\frac{1}{2}\alpha = \cos\frac{1}{2}\text{A}\cdots$, $\quad \sin\frac{1}{2}(\beta-\gamma) = -\sin\frac{1}{2}(\text{B}-\text{C})\cdots$,

and $\because$ the first members of these equations are positive,

$\therefore$ the second members are position.

NOTE 2. For any triangle, the equations

$$\sin\tfrac{1}{2}a/\sin\tfrac{1}{2}\alpha = \mp\sin\tfrac{1}{2}(b-c)/\sin\tfrac{1}{2}(\beta-\gamma),$$
$$\sin\tfrac{1}{2}b/\sin\tfrac{1}{2}\beta = \mp\sin\tfrac{1}{2}(c-a)/\sin\tfrac{1}{2}(\gamma-\alpha),$$
$$\sin\tfrac{1}{2}c/\sin\tfrac{1}{2}\gamma = \mp\sin\tfrac{1}{2}(a-b)/\sin\tfrac{1}{2}(\alpha-\beta),$$

must be taken all with the upper sign or all with the lower sign; and so of the other three groups of like equations.

For $\because \sin\frac{1}{2}a : \sin\frac{1}{2}\alpha = -\sin\frac{1}{2}(b-c) : \sin\frac{1}{2}(\beta-\gamma)$ [up. signs.

$$= \sin\tfrac{1}{2}a - \sin\tfrac{1}{2}(b-c) : \sin\tfrac{1}{2}\alpha + \sin\tfrac{1}{2}(\beta-\gamma)$$
$$= \sin\tfrac{1}{2}a + \sin\tfrac{1}{2}(b-c) : \sin\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}(\beta-\gamma),$$

$$\therefore \cos\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-b)/\sin\tfrac{1}{2}(\sigma-\gamma)\cos\tfrac{1}{2}(\sigma-\beta)$$
$$= \sin\tfrac{1}{2}(s-c)\cos\tfrac{1}{2}(s-b)/\cos\tfrac{1}{2}(\sigma-\gamma)\sin\tfrac{1}{2}(\sigma-\beta),$$

1] $\therefore \tan\frac{1}{2}(s-b)\cot\frac{1}{2}(s-c) = \cot\frac{1}{2}(\sigma-\beta)\tan\frac{1}{2}(\sigma-\gamma)$.

So, $\because \sin\frac{1}{2}a : \cos\frac{1}{2}\alpha = +\sin\frac{1}{2}(b+c) : \cos\frac{1}{2}(\beta-\gamma)$, [up. signs.

2] $\therefore \tan\frac{1}{2}s\cot\frac{1}{2}(s-a) = \cot\frac{1}{2}(\sigma-\beta)\cot\frac{1}{2}(\sigma-\gamma)$.

So, $\because \cos\frac{1}{2}a : \sin\frac{1}{2}\alpha = +\cos\frac{1}{2}(b-c) : \sin\frac{1}{2}(\beta+\gamma)$, [up. signs.

3] $\therefore \cot\frac{1}{2}(s-b)\cot\frac{1}{2}(s-c) = \tan\frac{1}{2}\alpha\cot\frac{1}{2}(\sigma-\alpha)$.

So, $\because \cos\frac{1}{2}a : \cos\frac{1}{2}\alpha = -\cos\frac{1}{2}(b+c) : \cos\frac{1}{2}(\beta+\gamma)$, [up. signs.

4] $\therefore \tan\frac{1}{2}s\tan\frac{1}{2}(s-a) = \cot\frac{1}{2}\alpha\cot\frac{1}{2}(\sigma-\alpha)$.

So, when the lower signs are taken :

5] $\cot\frac{1}{2}(s-b)\tan\frac{1}{2}(s-c) = \cot\frac{1}{2}(\sigma-\beta)\tan\frac{1}{2}(\sigma-\gamma)$,

6] $\cot\frac{1}{2}s\tan\frac{1}{2}(s-a) = \cot\frac{1}{2}(\sigma-\beta)\cot\frac{1}{2}(\sigma-\gamma)$,

7] $\tan\frac{1}{2}(s-b)\tan\frac{1}{2}(s-c) = \tan\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\alpha)$,

8] $\cot\frac{1}{2}s\cot\frac{1}{2}(s-a) = \cot\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\alpha)$;

and with a, α replaced by b, β, equations 5, 6, 7, 8 become :

9] $\cot\frac{1}{2}(s-a)\tan\frac{1}{2}(s-c) = \cot\frac{1}{2}(\sigma-\alpha)\tan\frac{1}{2}(\sigma-\gamma)$,

10] $\cot\frac{1}{2}s\tan\frac{1}{2}(s-b) = \cot\frac{1}{2}(\sigma-\alpha)\cot\frac{1}{2}(\sigma-\gamma)$,

11] $\tan\frac{1}{2}(s-a)\tan\frac{1}{2}(s-c) = \tan\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\beta)$,

12] $\cot\frac{1}{2}s\cot\frac{1}{2}(s-b) = \cot\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\beta)$,

with like formulæ if a, α be replaced by c, γ ;

and $\because$ the products of the equations 2, 4, and of 10, 12,

i.e. $\tan^2\frac{1}{2}s = \cot\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\alpha)\cot\frac{1}{2}(\sigma-\beta)\cot\frac{1}{2}(\sigma-\gamma)$,

and $\cot^2\frac{1}{2}s = \cot\frac{1}{2}\sigma\cot\frac{1}{2}(\sigma-\alpha)\cot\frac{1}{2}(\sigma-\beta)\cot\frac{1}{2}(\sigma-\gamma)$,

are contradictory, and so of other pairs of equations,

$\therefore$ the upper signs may not be used with the lower signs,

i.e. the upper signs must be used together and the lower signs together ;

and so for the other three groups of like equations ;

$\therefore$ with the entire set of twelve equations, the upper signs must be used together and the lower signs together.

NOTE 3. Another proof of Delambre's formulæ is as follows :

For $\sin\frac{1}{2}\alpha\cdot\sin\frac{1}{2}\beta = \sqrt{[\sin s\sin(s-a)/\sin b\sin c}$

$\cdot\sin s\sin(s-b)/\sin c\sin a]$

$= \pm\sin s/\sin c\cdot\sqrt{[\sin(s-a)\sin(s-b)/\sin a\sin b]}$

$= \pm\sin s/\sin c\cdot\cos\frac{1}{2}\gamma$.

So, $\sin\frac{1}{2}\alpha \cos\frac{1}{2}\beta = \pm\sin(s-a)/\sin c\cdot\sin\frac{1}{2}\gamma$,

$\cos\frac{1}{2}\alpha \sin\frac{1}{2}\beta = \pm\sin(s-b)/\sin c\cdot\sin\frac{1}{2}\gamma$,

$\cos\frac{1}{2}\alpha \cos\frac{1}{2}\beta = \pm\sin(s-c)/\sin c\cdot\cos\frac{1}{2}\gamma$.

In these equations the upper signs go together, and the lower signs go together.

For, multiply together the first two equations,

then $\sin^2\frac{1}{2}\alpha\cdot\sin\beta = \pm\sin s\sin(s-a)/\sin^2 c\cdot\sin\gamma$
$= \pm\sin^2\frac{1}{2}\alpha\cdot\sin b\sin c\sin\gamma/\sin^2 c$,

$\therefore \sin\beta = \pm\sin b\sin\gamma/\sin c$,

$\therefore \sin\beta/\sin b = \pm\sin\gamma/\sin c$;

and this equation is true only when the sign + is used,

i.e. when the two upper signs are taken in the first two equations, or the two lower signs;

and so of the other equations.

Add the first and fourth equations,

then $\cos\frac{1}{2}(\alpha-\beta) = [\sin s + \sin(s-c)]/\sin c\cdot\cos\frac{1}{2}\gamma$
$= \sin\frac{1}{2}(a+b)/\sin\frac{1}{2}c\cdot\cos\frac{1}{2}\gamma$;

and so for the other formulæ.

NOTE 4. Simon l'Huillier's formulæ. If equations 2, 3 of note 2 be multiplied together, the products give the equation

$$\tan\tfrac{1}{2}s\cot\tfrac{1}{2}(s-a)\cot\tfrac{1}{2}(s-b)\cot\tfrac{1}{2}(s-c)$$
$$=\tan\tfrac{1}{2}\sigma\cot\tfrac{1}{2}(\sigma-\alpha)\cot\tfrac{1}{2}(\sigma-\beta)\cot\tfrac{1}{2}(\sigma-\gamma);$$

and, by substitution from other equations of the set,

$$=\cot\tfrac{1}{2}(\sigma-\beta)\cot\tfrac{1}{2}(\sigma-\gamma)\cot\tfrac{1}{2}(s-b)\cot\tfrac{1}{2}(s-c),$$
$$=\cot\tfrac{1}{2}(\sigma-\gamma)\cot\tfrac{1}{2}(\sigma-\alpha)\cot\tfrac{1}{2}(s-c)\cot\tfrac{1}{2}(s-a),$$
$$=\cot\tfrac{1}{2}(\sigma-\alpha)\cot\tfrac{1}{2}(\sigma-\beta)\cot\tfrac{1}{2}(s-a)\cot\tfrac{1}{2}(s-b),$$
$$=\tan\tfrac{1}{2}\sigma\cot\tfrac{1}{2}(\sigma-\alpha)\tan\tfrac{1}{2}s\cot\tfrac{1}{2}(s-a),$$
$$=\tan\tfrac{1}{2}\sigma\cot\tfrac{1}{2}(\sigma-\beta)\tan\tfrac{1}{2}s\cot\tfrac{1}{2}(s-b),$$
$$=\tan\tfrac{1}{2}\sigma\cot\tfrac{1}{2}(\sigma-\gamma)\tan\tfrac{1}{2}s\cot\tfrac{1}{2}(s-c),$$

$$\therefore \tan\tfrac{1}{2}\sigma\tan\tfrac{1}{2}s = \cot\tfrac{1}{2}(\sigma-\alpha)\cot\tfrac{1}{2}(s-a),$$
$$=\cot\tfrac{1}{2}(\sigma-\beta)\cot\tfrac{1}{2}(s-b),$$
$$=\cot\tfrac{1}{2}(\sigma-\gamma)\cot\tfrac{1}{2}(s-c).$$

§ 11. THE RIGHT TRIEDRAL.

THEOR. 13. *In a triedral angle whose parts are $a, b, c, \alpha, \beta, \gamma$, if γ be a right diedral, then:*

$$\sin a = \sin c \sin \alpha = -\tan b \cot \beta,$$
$$\sin b = \sin c \sin \beta = -\tan a \cot \alpha,$$
$$\cos c = \cos a \cos b = \cot \alpha \cot \beta,$$
$$\cos \alpha = -\cos \alpha \sin \beta = -\tan b \cot c,$$
$$\cos \beta = -\cos b \sin \alpha = -\tan a \cot c.$$

For $\because \gamma$ is a right diedral,

$\therefore \sin \gamma = 0, \quad \cos \gamma = 0;$

and $\because \sin a/\sin \alpha = \sin b/\sin \beta = \sin c/\sin \gamma$, [theor. 6.

$\therefore \sin a/\sin \alpha = \sin c$,

and $\sin b/\sin \beta = \sin c$.

So, $\because \cos c = \cos a \cos b - \sin a \sin b \cos \gamma$, [theor. 5.

$\therefore \cos c = \cos a \cos b$.

So, $\because \cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a$, [theor. 5.

$\therefore \cos \alpha = -\sin \beta \cos a$.

So, $\because \cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b$, [theor. 5.

$\therefore \cos \beta = -\sin \alpha \cos b$.

So, $\because \cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c$, [theor. 5.

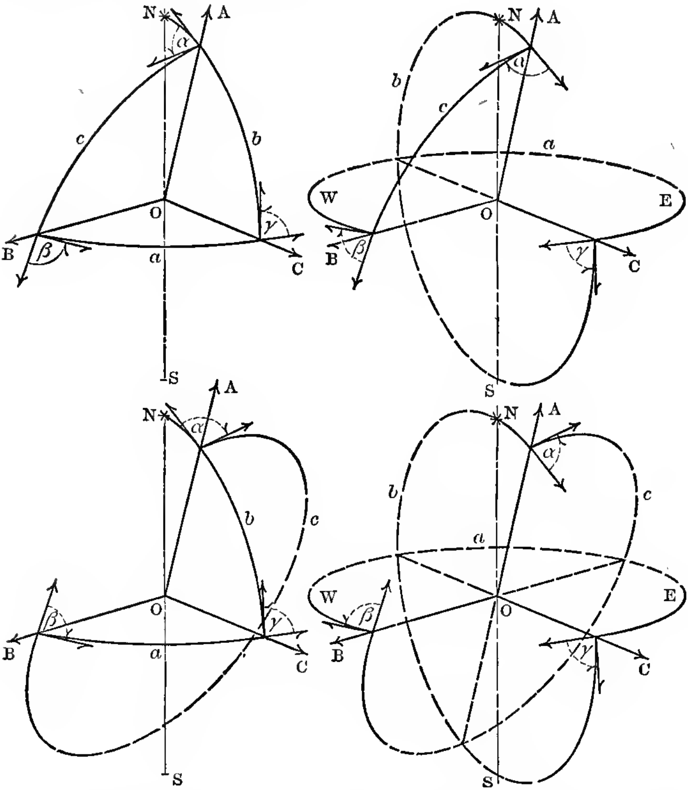
$\therefore \cos c = \cot \alpha \cot \beta$.

The four formulæ below come from the six proved above.

$$\cos \alpha = -\sin \beta \cos a = -\sin b/\sin c \cdot \cos c/\cos b = -\tan b \cot c,$$
$$\cos \beta = -\sin \alpha \cos b = -\sin a/\sin c \cdot \cos c/\cos a = -\tan a \cot c,$$
$$\sin a = \sin c \sin \alpha = -\sin b/\sin \beta \cdot \cos \beta/\cos b = -\tan b \cot \beta,$$
$$\sin b = \sin c \sin \beta = -\sin a/\sin \alpha \cdot \cos \alpha/\cos a = -\tan a \cot \alpha.$$

In the figures below, the great circle a may stand for the earth's equator, as it would be if the earth revolved from east to west half the time; the letters N, S stand for the north and south poles, the great circle b for a meridian, and the great circle c for any other great circle cutting the equator and the meridian. The angle γ is always a positive right angle. The

triangles appear as seen from the sun when fifteen degrees north of the equator. The invisible parts of the arcs are shown by the broken lines.



COR. *If* a, b, c, α, β, γ *be all positive and less than two right angles,* γ *a right diedral, and* A, B, C *the interior diedrals;*

then
$$\sin a = \sin c \sin \mathrm{A} = \tan b \cot \mathrm{B},$$
$$\sin b = \sin c \sin \mathrm{B} = \tan a \cot \mathrm{A},$$
$$\cos c = \cos a \cos b = \cot \mathrm{A} \cot \mathrm{B},$$
$$\cos \mathrm{A} = \cos a \sin \mathrm{B} = \tan b \cot c,$$
$$\cos \mathrm{B} = \cos b \sin \mathrm{A} = \tan a \cot c.$$

NAPIER'S RULES.

By an ingenious device of Lord Napier these ten formulæ are remembered by two simple rules:

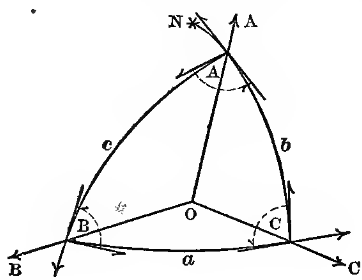
Ignore the right angle; take the two sides, and replace the hypotenuse and two oblique angles by their complements;

of the five parts so found call any one the middle part, *the two lying next it* adjacent parts, *and the others* opposite parts;

then: s*i*n m*i*d-part = prod t*a*n *a*dj*a* parts,

s*i*n m*i*d-part = prod c*o*s *o*pp*o* parts.

If the three parts considered lie together, that which lies between the other two is mid-part and the other two are adjacent parts; if two lie together and the third apart from them, the third one is mid-part and the other two are opposite parts.



E.g. let O-ABC be an ideal triedral right angled at C;

then, of the three parts a, b, co-c, co-c is mid-part, a, b are opposite parts,

and $\cos c = \cos a \cos b$.

So, of the parts co-A, co-B, co-c, co-c is mid-part, co-A, co-B are adjacent parts,

and $\cos c = \cot \text{A} \cot \text{B}$.

So, of the three parts co-A, co-B, a, co-A is mid-part, co-B, a are opposite parts,

and $\cos \text{A} = \sin \text{B} \cos a$.

§ 12. THE IDEAL TRIEDRAL.

A triedral whose face angles and diedrals are all positive and less than two right angles is an *ideal triedral.* Two parts of such a triedral are of the *same species* if they be both acute, both right, or both obtuse.

It has been shown in geometry that in an ideal triedral:

1. The sum of the three face angles lies between naught and four right angles.

2. The sum of the three interior diedrals lies between two right angles and six right angles.

3. Each face angle is less than the sum of the other two face angles, and so of the exterior diedrals.

4. Each interior diedral is greater than the difference between two right angles and the sum of the other two interior diedrals.

5. Of two unequal face angles the greater lies opposite the greater interior diedral, and so opposite the less exterior diedral, and conversely.

6. If two face angles be equal, so are the opposite diedrals (both exterior and interior), and conversely.

7. A plane through the vertical edge of an isosceles triangle perpendicular to the opposite face, bisects the interior vertical diedral and the opposite face angle.

Certain other facts relating to ideal triedrals are manifest, and still others appear by examining formulæ already proved.

8. The sine of every part is positive.

9. Every half part is positive and acute, and all its ratios are positive.

10. The half sum of two parts is positive and less than two right angles.

11. The half difference of two parts is acute and its cosine is positive.

THE IDEAL TRIANGLE.

THEOR. 14. *In an ideal triangle, that one of two unequal sides which is nearer right lies opposite the angle which is nearer right, and conversely.*

For $\because$ that angle which lies nearest to a right angle has the greatest sine,

and $\because \sin a/\sin A = \sin b/\sin B = \sin c/\sin C$,

$\therefore$ if $\sin a > \sin b$, then also $\sin A > \sin B$; and so of the others.

THEOR. 15. *In an ideal triangle, a side and its opposite interior angle are of the same species, if another side be as near right as the given side, or if another angle be as near right as the given angle.*

For let the side c be as near right as the given side a;

then $\because \cos c \not> \cos a$, and $\cos b < 1$,

$\therefore \cos b \cos c < \cos a$, or $\cos b \cos c = \cos a = 0$,

$\therefore \cos a$, $\cos a - \cos b \cos c$, are positive, negative, or zero together;

and $\because \cos A = (\cos a - \cos b \cos c)/\sin b \sin c$,

and $\sin b$, $\sin c$ are positive,

$\therefore \cos a$, $\cos A$ are positive, negative, or zero together,

$\therefore a$, A are both acute, both obtuse, or both right. Q.E.D.

So, let the angle C be as near right as the angle A;

then $\because \cos a = (\cos A + \cos B \cos C)/\sin B \sin C$,

and $\cos A$, $\cos A + \cos B \cos C$ are positive, negative, or zero together, [as above.

$\therefore \cos a$, $\cos A$ are positive, negative, or zero together,

$\therefore a$, A are both acute, both obtuse, or both right. Q.E.D.

COR. *A side and the opposite exterior angle may be both right; but if one of them be acute the other is obtuse, if another side be as near right as a given side, or another angle be as near right as a given angle.*

THEOR. 16. *In an ideal triangle the half-sum of two sides and the half-sum of their opposite interior angles are of the same species.*

For $\because \cos\frac{1}{2}(b+c)/\cos\frac{1}{2}(B+C)=\cos\frac{1}{2}a/\sin\frac{1}{2}A$,

and $\cos\frac{1}{2}a$, $\sin\frac{1}{2}A$ are both positive,

$\therefore \cos\frac{1}{2}(b+c)$, $\cos\frac{1}{2}(B+C)$ are positive, negative, or zero together,

$\therefore \frac{1}{2}(b+c)$, $\frac{1}{2}(B+C)$ are both acute, both obtuse, or both right. Q. E. D.

COR. *The half-sum of two sides and the half-sum of their two opposite exterior angles may be both right; but if one of them be acute the other is obtuse.*

§ 13. IDEAL RIGHT TRIANGLES.

A triangle having two right angles and two right sides is a *biquadrantal triangle.*

THEOR. 17. *In an ideal right triangle, if another part besides the right angle be right the triangle is biquadrantal.*

For let C be the right angle in the triangle ABC;

then $\because \cos c=\cos a\cos b$,

$\cos A=\cos a\sin B$,

$\cos B=\cos b\sin A$,

and $\sin A, \sin B\neq 0$, [A, B are not zero nor two right angles.

$\therefore$ if a be right, then $\cos a=0$,

$\therefore \cos c, \cos A=0$, and c, A are both right. Q. E. D.

So, if b be right, then c, B are right. Q. E. D.

So, if A be right, then $\cos A=0$,

$\therefore \cos a, \cos c=0$, and a, c are both right. Q. E. D.

So, if B be right, then b, c are right. Q. E. D.

So, if c be right, then $\cos c=0$,

$\therefore$ either $\cos a=0$, or $\cos b=0$,

$\therefore a$ or b is right, and A or B is right. Q. E. D.

THEOR. 18. *In an ideal right triangle, not biquadrantal, the hypotenuse is nearer right than either oblique side.*

For let C be the right angle in the triangle ABC;

then $\because \cos c = \cos a \cos b, \quad \cos a, \cos b < 1,$

$\therefore \cos c < \cos a, \quad \cos c < \cos b,$

$\therefore c$ is nearer right than a or b.

THEOR. 19. *In an ideal right triangle, not biquadrantal, an oblique angle is nearer right than its opposite side.*

For let C be the right angle in the triangle ABC;

then $\because \cos A = \cos a \sin B, \quad$ and $\quad \sin B < 1,$ [B not right.

$\therefore \cos A < \cos a,$

$\therefore$ A is nearer right than a. Q.E.D.

THEOR. 20. *In an ideal right triangle, an oblique angle and its opposite side are of the same species.*

For let C be the right angle in the triangle ABC;

then $\because \cos A = \cos a \sin B, \quad$ and $\quad \sin B$ is positive,

$\therefore \cos A, \quad \cos a$ are positive, negative, or zero together,

$\therefore$ A, a are both acute, both obtuse, or both right. Q.E.D.

THEOR. 21. *In an ideal right triangle, if the hypotenuse be acute the two oblique sides are of the same species, and so are the two oblique angles; but they are of opposite species if the hypotenuse be obtuse.*

For let C be the right angle in the triangle ABC;

then $\because \cos c = \cos a \cos b = \cot A \cot B,$

and $\quad \cos c$ is positive if c be acute,

$\therefore \cos a, \quad \cos b$ are both positive or both negative, and so are $\cot A, \quad \cot B$;

$\therefore a, b$ are both acute or both obtuse, and so are A, B.

So, $\because \cos c$ is negative if c be obtuse,

$\therefore \cos a, \cos b$ are of opposite signs, and so are $\cot A, \cot B$;

$\therefore a, b$ are of opposite species, and so are A, B.

QUESTIONS.

1. $\sin^2 \frac{1}{2}c = \sin^2 \frac{1}{2}a \cos^2 \frac{1}{2}b + \cos^2 \frac{1}{2}a \sin^2 \frac{1}{2}b$.
2. $\tan^2 \frac{1}{2}a = \tan \frac{1}{2}(c+b) \tan \frac{1}{2}(c-b)$.
3. $\tan^2 \frac{1}{2}\mathrm{A} = \sin(c-b)/\sin(c+b)$.
4. $\tan \frac{1}{2}\mathrm{A} = \sin(c-b)/\sin a \cos b = \sin a \cos b/\sin(c+b)$.
5. $\sin(a-b) = \sin a \tan \frac{1}{2}\mathrm{A} - \sin b \tan \frac{1}{2}\mathrm{B}$.
6. $\tan^2 \frac{1}{2}a = \tan \frac{1}{2}(\mathrm{B}+\mathrm{A}-\mathrm{R})/\tan \frac{1}{2}(\mathrm{B}-\mathrm{A}+\mathrm{R})$. [R a rt. ang.
7. $\tan^2 \frac{1}{2}c = -\cos(\mathrm{A}+\mathrm{B})/\cos(\mathrm{A}-\mathrm{B})$.

§ 14. SOLUTION OF IDEAL RIGHT TRIANGLES.

PROB. 2. TO SOLVE AN IDEAL RIGHT TRIANGLE, GIVEN TWO PARTS BESIDES THE RIGHT ANGLE.

Out of the formulæ of theor. 13, cor. *select those which involve the two given parts and one of the parts sought, and solve these three equations for the three parts sought.*

CHECK : *Substitute the three computed parts in that formula which involves them all, and see if they give an identity.*

Or, better, following Napier's rules :

Take each of the two given parts in turn for middle part, and apply that formula which brings in the other given part;

take the remaining part for middle part, and apply that formula which brings in both of the parts just found ;

solve the three equations so found for the parts sought.

CHECK : *Make the part last found the middle part, and apply that formula which brings in both the given parts.*

The check is defective in this, that it tests the logarithms, but not the angles got from these logarithms, *i.e.* not the final results : more perfect checks are got from any of the general formulæ which involve the three computed parts and one or more of the given parts.

The check is applied to the sine of the part last found ; if the two values got for this sine, natural or logarithmic, differ by not more than three units in the last decimal place, the work

is probably right, since the defects of the tables permit this discrepancy in the two results: if such discrepancy exist, the mean of the two values may be used.

These rules involve only the interior angles: for the general solution of the right triangle the formulæ of theor. 13 are available.

There are six cases.

(*a*) *Given a, b, the two sides about the right angle* C:

$$\begin{aligned} &\text{then} \because \sin a = \tan b \cot B, && \therefore \cot B = \sin a \cot b,\\ &\sin b = \tan a \cot A, && \therefore \cot A = \sin b \cot a,\\ &\cos c = \cot A \cot B; && \textit{check:}\ \cos c = \cos a \cos b. \end{aligned}$$

One triangle is always possible and but one: for the species of A, B, C is shown by the algebraic signs of the cotangents or cosines that are used, or by theors. 20, 21.

Geometrically. With the given arcs a, b at right angles, their extremities can be joined by a positive great arc, always in one way and in but one way.

(*b*) *Given c, a, the hypotenuse and one side:*

$$\begin{aligned} &\text{then}\quad \cos c = \cos a \cos b, && \therefore \cos b = \cos c/\cos a,\\ &\sin a = \sin c \sin A, && \therefore \sin A = \sin a/\sin c,\\ &\cos B = \cos b \sin A; && \textit{check:}\ \cos B = \tan a \cot c. \end{aligned}$$

A triangle is possible only when c is nearer right than a, or when c, a are both right.

The species of b, B is shown by the sign of cos b, cos B, and A is of the same species as a.

If c, a be both right, then cos b, cos B are indeterminate, b, B are equal, and A is right.

Geometrically. On a directed great circle b take any point C; through C draw a great circle a perpendicular to b, and take B a point on a such that BC is positive and equal to the given arc a; with B as pole, and an arc-radius equal to the given arc c, draw a small circle cutting the great circle b in two points; take A one of these points such that CA is positive:

then the triangle ABC is the triangle sought, and there is but one such triangle.

If the arc c be not nearer right than the arc a, the small circle is wholly within or wholly without the great circle b and there is no triangle.

(c) *Given* A, b, *an oblique angle and the adjacent side:*

then $\sin b = \tan a \cot \mathrm{A}$, $\quad\therefore \tan a = \sin b \tan \mathrm{A}$,

$\cos \mathrm{A} = \tan b \cot c$, $\quad\therefore \cot c = \cos \mathrm{A} \cot b$,

$\cos \mathrm{B} = \tan a \cot c$; *check:* $\cos \mathrm{B} = \cos b \sin \mathrm{A}$.

One triangle is always possible and but one. The species of the computed parts are shown by the signs of the tangent, cotangent, and cosine that are used, or by theors. 20, 21.

Geometrically. On a directed great circle b, take C, A two points such that the arc CA is positive and equal to the given arc B; at C draw the directed great circle a, perpendicular to the circle b, and at A draw the directed great circle c making an angle with the circle b equal to α, the supplement of the given angle A, and meeting the great circle a in two points; take B one of these points such that BC, CA are positive arcs: then the triangle ABC is the triangle sought, and there is but one triangle.

(d) *Given* B, b, *an oblique and its opposite side:*

then $\cos \mathrm{B} = \cos b \sin \mathrm{A}$, $\quad\therefore \sin \mathrm{A} = \cos \mathrm{B}/\cos b$,

$\sin b = \sin c \sin \mathrm{B}$, $\quad\therefore \sin c = \sin b/\sin \mathrm{B}$,

$\sin a = \sin \mathrm{A} \sin c$; *check:* $\sin a = \tan b \cot \mathrm{B}$.

If B, b be not of the same species, no triangle is possible, for then $\sin \mathrm{A}$ is negative.

If b be nearer right than B, no triangle is possible, for then $\sin \mathrm{A} > 1$, which is impossible.

If B, b be equal but not right, the triangle is biquadrantal, for then $\sin \mathrm{A}$, $\sin a$, $\sin c$ are all 1, and A, a, c are all right.

If B, b be both right, the triangle is biquadrantal, for then c also is right, and A, a are indeterminate.

If B be nearer right than b and of the same species, there are two triangles.

For $\because$ A, a, c are all found from their sines,

and sin A, sin a, sin c are all positive,

$\therefore$ to each of these sines correspond two possible angles, supplementary to each other, both positive and less than two right angles.

But A, a must be of the same species;

and if c be acute, A, a are of the same species with B, b.

So, if c be obtuse, A, a are of species opposite to that of B, b,

$\therefore$ two triangles, and but two, are possible.

Geometrically. Let BDB′, B′EB be half circles forming lines whose angle is the given angle B of the triangle; draw an arc CA normal to BCB′ and equal to the given arc b; with its initial point C sliding over BCB′ push the arc-normal CA to the right and left till the terminal point A rests on the circle B′EB;

then if B be acute and $b<$ B, two triangles A′BC′, A″BC″;

if $b=$ B, one triangle, biquadrantal; if $b>$ B, no triangle.

So, if B be obtuse and $b>$ B, two triangles are formed;

if $b=$ B, one triangle, biquadrantal; if $b<$ B, no triangle.

(*e*) *Given c*, A, *the hypotenuse and an oblique angle:*

then $\cos c = \cot A \cot B$, $\quad \therefore \cot B = \cos c \tan A$,

$\cos A = \tan b \cot c$, $\quad \therefore \tan b = \tan c \cos A$,

$\sin a = \tan b \cot B$; *check:* $\sin a = \sin c \sin A$.

One triangle is always possible and but one, for the species of b, B is shown by the sign of the tangent and cotangent, and a, A are of the same species.

Geometrically. Through a point A on a great circle b, draw the great circle c, making with the great circle b the given angle A; lay off AB positive and equal to the given arc c; from B draw the great circle a perpendicular to the circle b and meeting it in C; then is the triangle ABC the triangle sought, and this triangle can be drawn in but one way.

(*f*) *Given* A, B *the two oblique angles:*

then $\cos A = \cos a \sin B$, $\therefore \cos a = \cos A/\sin B$,

$\cos B = \cos b \sin A$, $\therefore \cos b = \cos B/\sin A$,

$\cos c = \cos a \cos b$; *check:* $\cos c = \cot A \cdot \cot B$.

The species of the parts sought is shown by the signs of the cosines; but the solution is possible only when $\cos A < \sin B$ and $\cos B < \sin A$,

i.e. when A is nearer right than co-B, and B than co-A.

Geometrically. Let AD, A′E be great circles whose angle is the given angle A, and draw great arcs CB, C′B′, C″B″, · · ·, normal to the arc ADA′; then the angles CBA vary from R − A to R + A, and one of them is the given angle B, if B be nearer right than co-A, and but one.

QUADRANTAL TRIANGLES.

Find the polar of the given triangle: it is a right triangle; solve this triangle and take the supplements of the parts thus found for the parts sought in the given triangle.

A biquadrantal triangle is indeterminate unless either the base or the vertical angle be given.

ISOSCELES TRIANGLES.

Draw an arc from the vertex to the middle of the base, thereby dividing the given triangle into two equal right triangles; solve one of these triangles.

If only the base and the vertical angle be given, there are two triangles, one triangle, or none, according as the base is less than, equal to, or greater than the vertical angle; if only the two equal sides or the two equal angles be given, there is an infinite number of triangles; otherwise, subject to the conditions just found (*b*, *f*), there is one triangle, and but one.

OBLIQUE TRIANGLES.

In most cases a perpendicular may fall from a vertex of an oblique triangle to the opposite side in such manner that one

of the right triangles thus formed contains two of the three known parts, and the other right triangle contains one of them.

Solve these right triangles in order and so combine the parts as to find the parts sought in the given triangle.

The solution of right triangles may take this form:

Given $a=72°$, $b=125°$, $\mathrm{C}=90°$, then:

$\cot \mathrm{A}=\cot a \sin b$, $\cot \mathrm{B}=\sin a\ \overline{\cot}\, b$,

9.511776	9.978206
9.913365	9.845227
9.425141	9.823433 neg.
$\mathrm{A}=75°\ 5'\ 45''$.	$\mathrm{B}=123°\ 39'\ 40''$. [theor. 20.

$\cos c=\cot \mathrm{A}\ \overline{\cot}\, \mathrm{B}$, *check:* $\cos c=\cos a\ \overline{\cos}\, b$.

9.425141	9.489982
9.823433	9.758591
9.248574 neg.	9.248573 neg.
$c=100°\ 12'\ 34''$.	[theor. 21.

QUESTIONS.

Solve these right triangles, given C, a right angle, and:

1. a, 116°; b, 16°. [97° 39′ 24″, 17° 41′ 40″, 114° 55′ 20″.
2. c, 140°; a, 20°. [32° 8′ 48″, 115° 42′ 24″, 144° 36′ 29″.
3. A, 80° 10′; b, 155°. [67° 6′ 23″, 153° 57′ 34″, 110° 46′ 40″.
4. A, 100°; a, 112°. [27° 36′ 59″, 109° 41′ 49″, 25° 52′ 33″, or 152° 23′ 1″, 70° 18′ 11″, 154° 7′ 27″.
5. c, 120°; A, 120°. [131° 24′ 34″, 40° 53′ 36″, 49° 6′ 24″.
6. A, 60° 47′; B, 57° 16′. [54° 31′ 52″, 51° 43′ 1″, 68° 55′ 50″.
7. c, 140°; a, 140°.
8. c, 120°; A, 90°.

Solve these quadrantal triangles, given:

9. A, 80°; a, 90°; b, 37°.
10. B, 50°; b, 130°; c, 90°.

Solve these isosceles triangles, given:

11. a, 70°; b, 70°; A, 30°. [157° 39′ 34″, 134° 24′ 30″.
12. a, 30°; A, 70°; B, 70°.
13. a, 119°; b, 119°; C, 85°. [113° 57′ 11″, 72° 26′ 22″.

§ 15. SOLUTION OF IDEAL OBLIQUE TRIANGLES.

PROB. 3. TO SOLVE AN IDEAL OBLIQUE TRIANGLE, GIVEN ANY THREE PARTS.

Apply such of the formulæ of theor. 5 *cor.* 4, *theor.* 6 *cor., theor.* 11 *cor., as serve to express the three parts sought in terms of known parts.*

Where possible the computer will choose his formulæ so as to avoid angles near the ends of a quarter.

CHECK: *form an equation that involves the three computed parts and such of the given parts as may be necessary.*

If the equation so formed be a true equation the parts have probably been computed correctly.

Delambre's formulæ are useful as checks, and so are the formulæ shown in the questions in § 13.

In the check the parts must be involved by different ratios, or in different combinations, from those used in the solution.

NOTE. These rules involve only the interior angles : for the general solution of the oblique triangle, the formulæ of theors. 5, 6, 11 and their corollaries are available.

There are six cases :

(*a*) *Given b, c,* A, *two sides and the included angle:*

then $\tan\frac{1}{2}(\mathrm{B}+\mathrm{C})=\cot\frac{1}{2}\mathrm{A}\cdot\cos\frac{1}{2}(b-c)/\cos\frac{1}{2}(b+c)$,

$\tan\frac{1}{2}(\mathrm{B}-\mathrm{C})=\cot\frac{1}{2}\mathrm{A}\cdot\sin\frac{1}{2}(b-c)/\sin\frac{1}{2}(b+c)$,

$\mathrm{B}=\frac{1}{2}(\mathrm{B}+\mathrm{C})+\frac{1}{2}(\mathrm{B}-\mathrm{C}),\quad \mathrm{C}=\frac{1}{2}(\mathrm{B}+\mathrm{C})-\frac{1}{2}(\mathrm{B}-\mathrm{C})$,

$\tan\frac{1}{2}a=\tan\frac{1}{2}(b+c)\cdot\cos\frac{1}{2}(\mathrm{B}+\mathrm{C})/\cos\frac{1}{2}(\mathrm{B}-\mathrm{C})$.

Check: the law of sines or one of Delambre's formulæ.

There is always one triangle and but one.

For ∵ whatever the values of *b, c,* A, the parts given,

$\tan\frac{1}{2}(\mathrm{B}+\mathrm{C})$, $\tan\frac{1}{2}(\mathrm{B}-\mathrm{C})$, $\tan\frac{1}{2}a$ are always possible,

and the species of $\frac{1}{2}(\mathrm{B}+\mathrm{C})$, $\frac{1}{2}(\mathrm{B}-\mathrm{C})$, $\frac{1}{2}a$ are shown by the signs of their tangents,

∴ B, C, *a* are always possible, and they have single values.

Geometrically. Lay off the arc CA equal to the given arc b; at A turn by the angle α, the supplement of A, and lay off the arc AB equal to the given arc c; join BC: the triangle ABC is the triangle sought, and with the data there is always one and but one such triangle.

(*b*) *Given* B, C, a, *two angles and their common side:*

then $\tan\frac{1}{2}(b+c)=\tan\frac{1}{2}a\cdot\cos\frac{1}{2}(B-C)/\cos\frac{1}{2}(B+C)$,

$\tan\frac{1}{2}(b-c)=\tan\frac{1}{2}a\cdot\sin\frac{1}{2}(B-C)/\sin\frac{1}{2}(B+C)$,

$b=\frac{1}{2}(b+c)+\frac{1}{2}(b-c), \quad c=\frac{1}{2}(b+c)-\frac{1}{2}(b-c)$,

$\cot\frac{1}{2}A=\tan\frac{1}{2}(B+C)\cdot\cos\frac{1}{2}(b+c)/\cos\frac{1}{2}(b-c)$.

Check: the law of sines, or one of Delambre's formulæ.

There is always one triangle and but one.

For $\because$ whatever the values of B, C, a, the parts given,

$\tan\frac{1}{2}(b+c)$, $\tan\frac{1}{2}(b-c)$, $\tan\frac{1}{2}A$ are always possible,

and the species of $\frac{1}{2}(b+c)$, $\frac{1}{2}(b-c)$, $\frac{1}{2}A$ are shown by the sign of their tangents,

$\therefore$ b, c, A are always possible, and they have single values.

Geometrically. At any point B, on an indefinite arc AB, turn by the angle β, the supplement of B, and lay off the arc BC equal to the given arc a; at C turn by the angle γ, the supplement of C, and draw an arc cutting AB in A: the triangle ABC is the triangle sought, and with the data there is always one and but one such triangle.

NOTE. This triangle may be solved under case (*a*), using the polar triangle whose parts b', c', A$'$ are supplementary to B, C, a, and the computed parts B$'$, C$'$, a', to b, c, A, the parts sought.

(*c*) *Given* b, c, B, *two sides and an angle opposite one of them:*

then $\sin C=\sin c\cdot\sin B/\sin b$,

$\cot\frac{1}{2}A=\tan\frac{1}{2}(B+C)\cos\frac{1}{2}(b+c)/\cos\frac{1}{2}(b-c)$,

$\tan\frac{1}{2}a=\tan\frac{1}{2}(b+c)\cos\frac{1}{2}(B+C)/\cos\frac{1}{2}(B-C)$.

Check: one of Delambre's formulæ.

If b, c, B be all right, the triangle is biquadrantal, and A, a are indeterminate and equal.

If $\sin c \sin B > \sin b$, then $\sin c > 1$, which is impossible, and there is no triangle.

If $\sin c \sin B = \sin b$, then $\sin C = 1$, C is right, and there is one (a right) triangle if b, B be of the same species, but no triangle if they be of opposite species.

If $\sin c \sin B < \sin b$, then $\sin C < 1$, and C may be either of two supplementary angles; but these angles must be taken subject to the law, the greater angle lies opposite the greater side.

In particular:

If c be nearer right than b, there are two triangles if b, B be of the same species, but none if they be of opposite species.

If c be just as near right as b, there is one (an isosceles) triangle if b, B be of the same species, but no triangle if they be of opposite species; and c, C are also of the same species.

If c be less near right than b, there is one triangle and c, C are of the same species.

Geometrically. Lay off an arc AB equal to the given side c; at B turn by the angle β, the supplement of B, and lay off the indefinite arc BC; with A as pole and an arc-radius equal to b describe a small circle:

if this small circle neither cuts nor touches the circle BC, there is no triangle;

if it touches the arc BC at a point C, such that BC is positive and less than two right angles, there is a right triangle;

if it cuts the arc BC at one point C, such that BC is a limited arc, there is one triangle;

if it cuts the arc BC in two points C_1, C_2, such that BC_1, BC_2 are both limited arcs, there are two triangles.

(*d*) *Given* B, C, b, *two angles and a side opposite one of them:*

then $\sin c = \sin C \cdot \sin b / \sin B$,

$$\tan \tfrac{1}{2} a = \tan \tfrac{1}{2}(b+c) \cos \tfrac{1}{2}(B+C) / \cos \tfrac{1}{2}(B-C),$$

$$\cot \tfrac{1}{2} A = \tan \tfrac{1}{2}(B+C) \cos \tfrac{1}{2}(b+c) / \cos \tfrac{1}{2}(b-c).$$

Check: one of Delambre's formulæ.

If B, C, b be all right, the triangle is biquadrantal, and a, A are indeterminate and equal.

If $\sin C \sin b > \sin B$ then $\sin c > 1$, which is impossible, and there is no triangle.

If $\sin C \sin b = \sin B$, then $\sin c = 1$, c is right and there is one (a quadrantal) triangle if B, b be of the same species, but no triangle if they be of opposite species.

If $\sin C \sin b < \sin B$, then $\sin c < 1$, and c may be either of two supplementary arcs; but these arcs must be taken subject to the law that in an ideal spherical triangle the greater angle lies opposite the greater side. In particular:

If C be nearer right than B, there are two triangles if B, b be of the same species, but none if they be of opposite species.

If C be just as near right as B, there is one (an isosceles) triangle if b, B be of the same species, but none if they be of opposite species. In this triangle c, C are also of the same species.

If C be less near right than B, there is one triangle, and c, C are of the same species.

Geometrically. Draw an indefinite arc; at any point B, turn by the angle β, the supplement of B, and draw the indefinite arc BC; at any point C, turn by the angle γ, the supplement of C, and draw the arc CA equal to b; let the arc first drawn slide along the arc BC, as a line may slide along one of its bounding circles, without changing the angle B.

If this sliding arc do not pass through the point A, there is no triangle; if it touch A, and the angle A be a limited angle, there is one (a quadrantal) triangle; if it pass through A twice, so as to make the two angles A_1, A_2 both limited angles, there are two triangles.

(*e*) *Given a, b, c, the three sides:* then

$$\tan \tfrac{1}{2}A = \sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)/\sin s]}/\sin(s-a),$$
$$\tan \tfrac{1}{2}B = \sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)/\sin s]}/\sin(s-b),$$
$$\tan \tfrac{1}{2}C = \sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)/\sin s]}/\sin(s-c).$$

Check: one of Delambre's formulæ.

Since each of the half angles is positive and acute, the radical must be taken positive and there is no ambiguity; but no triangle is possible unless s, $s-a$, $s-b$, $s-c$ be all positive.

(*f*) *Given* A, B, C, *the three angles:*

then $\tan \frac{1}{2}a = \sqrt{[\sin E/\sin (A-E) \sin (B-E) \sin (C-E)]} / \sin (A-E),$

$\tan \frac{1}{2}b = \sqrt{[\sin E/\sin (A-E) \sin (B-E) \sin (C-E)]} / \sin (B-E),$

$\tan \frac{1}{2}c = \sqrt{[\sin E/\sin (A-E) \sin (B-E) \sin (C-E)]} / \sin (C-E).$

Check: one of Delambre's formulæ.

Since each of the half sides is positive and acute, the radicals must be taken positive and there is no ambiguity; but no triangle is possible unless E, A − E, B − E, C − E be all positive.

QUESTIONS.

Solve these oblique triangles, given:

1. a, 100°; b, 50°; c, 60°.
[138° 15′ 45″, 31° 11′ 14″, 35° 49′ 58″.
2. A, 120°; B, 130°; C, 80°.
[144° 10′ 2″, 148° 48′ 46″, 41° 44′ 15″.
3. b, 98° 12′; c, 80° 35′; A, 10° 16′.
[149° 32′ 51″, 30° 20′ 29″, 20° 22′ 7″.
4. A, 135° 15′; C, 50° 30′; b, 69° 34′.
[50° 6′ 16″, 120° 41′ 47″, 70° 28′ 9″.
5. a, 40° 16′; b, 47° 14′; A, 52° 30′.
6. a, 120°; b, 70°; A, 130°.
[58° 57′ 20″, 75° 36′ 4″, 56° 13′ 23″,
or 165° 23′ 44″, 163° 26′ 16″, 123° 46′ 37″.
7. a, 40°; b, 50°; A, 50°.
[65° 54′ 52″, 82° 48′ 42″, 56° 21′ 24″,
or 114° 5′ 8″, 22° 16′ 52″, 18° 33′ 2″.
8. A, 132° 16′; B, 139° 44′; a, 127° 30′.
[136° 8′ 16″, 114° 17′ 48″, 77° 43′ 4″.
9. A, 110°; B, 60°; a, 50°.
10. A, 70°; B, 120°; a, 80°.
[114° 49′ 26″, 65° 48′ 58″, 72° 56′ 48″.

PROB. 4. TO FIND THE AREA OF AN IDEAL TRIANGLE.

It is shown in geometry that the area of an ideal spherical triangle bears the same ratio to the area of a trirectangular triangle as the spherical excess bears to a right angle: *i.e.* if ABC be a limited spherical triangle, and if K stand for the area of the triangle, T for the area of the trirectangular triangle, and 2E for the spherical excess $A+B+C-2R$, then

$$K:T=2E:R \quad \text{and} \quad K=T\cdot 2E/R.$$

This area may also be expressed in terms of a, b, c, the sides of the triangle, as follows:

Divide the equation [theor. 12, note 2, form. 4.

$$\tan\tfrac{1}{2}s\cdot\tan\tfrac{1}{2}(s-a)=\cot\tfrac{1}{2}\sigma\cdot\cot\tfrac{1}{2}(\sigma-\alpha)$$

by the equation [theor. 12, note 2, form. 3.

$$\cot\tfrac{1}{2}(s-b)\cot\tfrac{1}{2}(s-c)=\tan\tfrac{1}{2}\sigma\cot\tfrac{1}{2}(\sigma-\alpha),$$

then $\cot^2\tfrac{1}{2}\sigma=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)$;

and $\because 2E=(A+B+C)-2R=4R-(\alpha+\beta+\gamma)=4R-2\sigma$,

$$\therefore \tfrac{1}{2}E=R-\tfrac{1}{2}\sigma, \quad \text{and} \quad \tan\tfrac{1}{2}E=\cot\tfrac{1}{2}\sigma,$$

$$\therefore \tan^2\tfrac{1}{2}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c),$$

$$\therefore K=T\cdot 4[\tan^{-1}\sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}]/R.$$

Manifestly the radical has the positive sign for an ideal triangle, and the angle is the smallest positive angle in the group of congruent angles shown by the bracket.

§ 16. RELATIONS OF PLANE AND SPHERICAL TRIANGLES.

After the definition of the trigonometric ratios and the statement of their relations, all the properties of the right spherical triangle, and of the plane triangle (oblique and right), may be derived from those of the oblique spherical triangle. Such a development of the subject presents the principles of trigonometry in their most general form, and teaches the student to take these general propositions and, by successive steps, to draw out and state in their logical order the special propositions that are included in them. This mutual relation of the general and the particular not only helps the intellect to grasp these propositions, but also helps the memory to retain them.

The order of development is this :

to state and prove the general properties of the oblique spherical triangle, counting it the most general form of the triangle.

to derive the properties of the right spherical triangle, counting it a special case of the oblique spherical triangle wherein one angle is a right angle.

to derive the general properties of the oblique plane triangle, counting it a special case of the spherical triangle wherein the radius of the sphere has become infinite.

to derive the properties of the right plane triangle, counting it a special case of the oblique plane triangle wherein one of the angles is a right angle, or of the right spherical triangle wherein the arcs are straight lines.

The general properties of the plane triangle may be got from those of the spherical triangles as follows :

If the sides of a spherical triangle subtend very small angles at the centre of the sphere, the spherical triangle differs but little from a plane triangle having the same vertices ; and, if the vertices be fixed in position while the centre of the sphere recedes further and further away, and the radii grow longer and longer, then the bounding arcs grow straighter, the spherical triangle approaches closer to the plane triangle having the

same vertices, the small angles at the centre of the sphere subtended by the sides of the triangle are proportional to those sides, and the sum of the three angles of the triangle is a little greater than, but approaches, two right angles.

The plane triangle that has the same vertices is the limit to which the spherical triangle approaches when the radius is infinite, and if in the formulæ for spherical triangles the functions of the sides be expressed in terms of their subtended angles, and only those terms be retained whose limiting ratios are finite, *i.e.* those that are of the same lowest order of infinitesimal, the resulting formulæ correspond to the formulæ for plane triangles.

In detail: replace $\sin a$, $\cos a$, $\tan a \cdots$ by

$$a-a^3/3!+\cdots,\quad 1-a^2/2!+\cdots,\quad a+a^3/3+\cdots;$$

omit all terms except those of lowest order, and replace those infinitesimals by the corresponding sides of the plane triangle.

(*a*) *The terms of lowest degree of the first order:*

Replace $\sin a$, $\tan a$, $2\sin\frac{1}{2}a \cdots$ by a; $\cos a$ by 1; and so on; then $\because \sin a=\sin c\sin A=\tan b\cot B$,

$$\therefore a=c\sin A=b\cot B;$$

and $\because \cos A=\cos a\sin B=\tan b\cot c$,

$$\therefore \cos A=1\cdot\sin B=b/c;$$

and $\because \sin a/\sin b=\sin A/\sin B$,

$$\therefore a/b=\sin A/\sin B;$$

and $\because \sin\frac{1}{2}A=\sqrt{[\sin(s-b)\sin(s-c)/\sin b\sin c]}$,

$$\therefore \sin\tfrac{1}{2}A=\sqrt{[(s-b)(s-c)/bc]};$$

and $\because \cos\frac{1}{2}A=\sqrt{[\sin s\sin(s-a)/\sin b\sin c]}$,

$$\therefore \cos\tfrac{1}{2}A=\sqrt{[s(s-a)/bc]};$$

and $\because \tan\frac{1}{2}A=\sqrt{[\sin(s-b)\sin(s-c)/\sin s\sin(s-a)]}$,

$$\therefore \tan\tfrac{1}{2}A=\sqrt{[(s-b)(s-c)/s(s-a)]};$$

and $\because \sin\frac{1}{2}(A-B)/\cos\frac{1}{2}C=\sin\frac{1}{2}(a-b)/\sin\frac{1}{2}c$,

$$\therefore \sin\tfrac{1}{2}(A-B)/\cos\tfrac{1}{2}C=(a-b)/c;$$

and $\because \cos\frac{1}{2}(\text{A}-\text{B})/\sin\frac{1}{2}\text{C}=\sin\frac{1}{2}(a+b)/\sin\frac{1}{2}c$,

$\therefore \cos\frac{1}{2}(\text{A}-\text{B})/\sin\frac{1}{2}\text{C}=(a+b)/c$;

and $\because \tan\frac{1}{2}(\text{A}-\text{B})/\cot\frac{1}{2}\text{C}=\sin\frac{1}{2}(a-b)/\sin\frac{1}{2}(a+b)$,

$\therefore \tan\frac{1}{2}(\text{A}-\text{B})/\cot\frac{1}{2}\text{C}=(a-b)/(a+b)$;

and $\because \tan\frac{1}{2}(a+b)/\tan\frac{1}{2}c=\cos\frac{1}{2}(\text{A}-\text{B})/\cos\frac{1}{2}(\text{A}+\text{B})$,

$\therefore (a+b)/c=\cos\frac{1}{2}(\text{A}-\text{B})/\cos\frac{1}{2}(\text{A}+\text{B})$;

and $\because \tan\frac{1}{2}(a-b)/\tan\frac{1}{2}c=\sin\frac{1}{2}(\text{A}-\text{B})/\sin\frac{1}{2}(\text{A}+\text{B})$,

$\therefore (a-b)/c=\sin\frac{1}{2}(\text{A}-\text{B})/\sin\frac{1}{2}(\text{A}+\text{B})$.

(*b*) *The terms of lowest degree of the second order:*

Replace $\sin a$, $\tan a$, by a; $\cos a$ by $1-\frac{1}{2}a^2$; and so on. *E.g.* the formula $\cos c=\cos a \cos b$ becomes

$$1-\tfrac{1}{2}c^2+\cdots=(1-\tfrac{1}{2}a^2+\cdots)(1-\tfrac{1}{2}b^2+\cdots)$$
$$=1-\tfrac{1}{2}a^2-\tfrac{1}{2}b^2+\cdots;$$

$\therefore a^2+b^2=c^2\pm$ terms of higher degree, whose ratios to a^2, b^2, $c^2 \doteq 0$ when a, b, $c \doteq 0$,

$\therefore a^2+b^2=c^2$.

So, the formula $\cos a=\cos b \cos c+\sin b \sin c \cos \text{A}$ gives

$a^2=b^2+c^2-2bc\cos\text{A}$.

QUESTIONS.

1. Show that, for a plane right triangle, of exs. 1–7, § 13,
   exs. 1, 2 reduce to $a^2+b^2=c^2$;
   exs. 3, 4, to $\tan\frac{1}{2}\text{A}=\sqrt{[(c-b)/(c+b)]}=a/(c+b)$;
   ex. 5, to $a-b=a\tan\frac{1}{2}\text{A}-b\tan\frac{1}{2}\text{B}$;
   exs. 6, 7, to $\text{A}+\text{B}=90°$.
2. Show that, for a plane triangle, of exs. 1–12, § 7,
   ex. 1 reduces to ex. 4, III § 2;
   ex. 3, to ex. 7, III § 2;
   ex. 5, to ex. 5, III § 2;
   ex. 7, to $(s-c)/(s-a)=\tan\frac{1}{2}\text{A}/\tan\frac{1}{2}\text{C}$.
3. Show what the other examples of § 7 reduce to.

§ 17. LEGENDRE'S THEOREM.

THEOR. 22. *If* ABC *be any spherical triangle whose sides are very small as to the radius of the sphere, and if* A'B'C' *be a plane triangle whose sides* a', b', c' *are equal in absolute length to the sides* a, b, c *of the spherical triangle; then each angle* A, B, C *exceeds the corresponding angle* A', B', C' *by one-third of the spherical excess of the triangle* ABC.

For, replace $\sin b$, $\sin c$ by $b-\frac{1}{6}b^3+\cdots$, $c-\frac{1}{6}c^3+\cdots$, and $\cos a$, $\cos b$, $\cos c$ by $1-\frac{1}{2}a^2+\cdots$, $1-\frac{1}{2}b^2+\cdots$, $1-\frac{1}{2}c^2+\cdots$; then $\because$ the formula $\cos a=\cos b\cos c+\sin b\sin c\cos A$ gives

$$bc(\cos A'-\cos A) \qquad [\cos A'\equiv(b^2+c^2-a^2)/2bc.$$
$$=\tfrac{1}{12}(a^2b^2+b^2c^2+c^2a^2)-\tfrac{1}{24}(a^4+b^4+c^4)\pm \text{terms whose}$$
ratios to these terms $\doteq 0$, when a, b, $c\doteq 0$,

and so for $ca(\cos B'-\cos B)$, $ab(\cos C'-\cos C)$, [sym.

$$\therefore bc(\cos A'-\cos A)\doteq ca(\cos B'-\cos B)\doteq ab\cos C'-\cos C).$$

But $\because \cos A'-\cos A=2\sin\frac{1}{2}(A-A')\sin\frac{1}{2}(A+A')$

$$\doteq(A-A')\sin A',$$

and so for $\cos B'-\cos B$, $\cos C'-\cos C$; [sym.

$$\therefore bc(A-A')\sin A'\doteq ca(B-B')\sin B'\doteq ab(C-C')\sin C;$$

and $\because \sin A'$, $\sin B'$, $\sin C'$ are proportional to a, b, c,

$$\therefore A-A'\doteq B-B'\doteq C-C'\doteq\tfrac{1}{3}[(A+B+C)-(A'+B'+C')].$$

QUESTIONS.

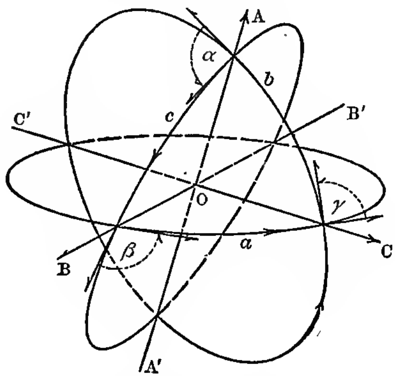
1. Triangles upon the earth's surface are regarded as spherical triangles, and the earth's mean radius is 3956 miles. If the angles A, B be 65°, 60° and the side c be 100 miles, find the sides a, b in degrees and in miles; find the angle C and the spherical excess; find the area of the triangle in square miles; find the number of square miles that correspond to 1″ of spherical excess.

2. In a geodetic survey the angles A, B, C are 48° 45′, 30°, 101° 15′ 12″, and the side c is 70 miles: find the angles of the plane triangle whose sides equal a, b, c, of the spherical triangle, and thence find the lengths of a, b. [Leg. th.

§ 18. THE GENERAL SPHERICAL TRIANGLE.

On page 116 it was shown that three lines, A′A, B′B, C′O, which meet in a point O, give four pairs of symmetric triedral angles, in the geometric sense :

O-ABC, O-A′B′C′; O-A′BC, O-AB′C′;
O-AB′C, O-A′BC′; O-ABC′, O-A′B′C;

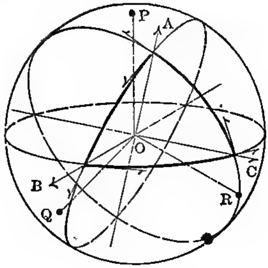


and that when the planes of these lines are properly directed, each triedral gives eight spherical triangles, thus forming eight groups of eight, each of the sixty-four differing in some way from every other one of them.

So, it was shown on page 124 that the law of cosines and the law of sines hold true for all spherical triangles and all triedrals, without regard to the signs or magnitudes of their parts ; and consequently that all the formulæ derived from these laws hold true universally.

In these triangles the circuit is so made that the vertices are always taken in the order A-B-C-A-B, never in the order A-C-B-A-C : that order would give sixty-four more triangles.

It remains to show how, starting with one of them, *e.g.* the ideal triangle, the other sixty-three triangles may be derived from it ; and how, in the solution of the general spherical triangle, the species of the parts may be known.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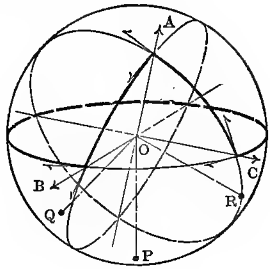
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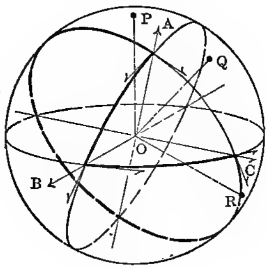
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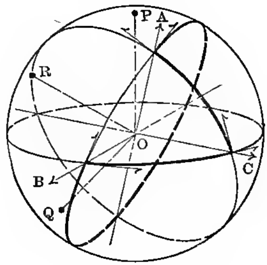
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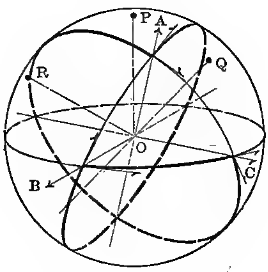
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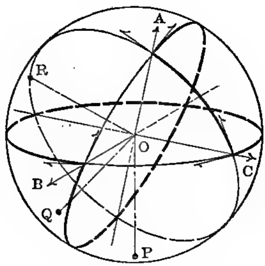
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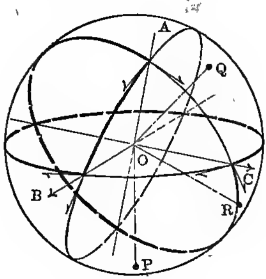
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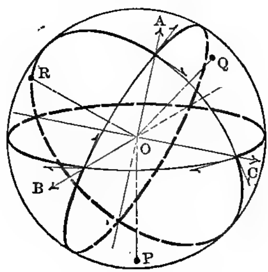
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C
B
P

THE DEFORMATION OF SPHERICAL TRIANGLES.

Let ABC be an ideal spherical triangle, as shown in the first figure, with the parts a, b, c, α, β, γ named and used as on page 112;

let the plane a turn upon one of its own diameter, *e.g.* on B′B or on C′C, as upon a hinge, till it comes again to pass through the points B, C, but has its direction reversed, as in the second figure;

then, while the arcs b, c are unchanged, the arc a is replaced by its explement, $2\pi - a$;

and ∵ OP, the axis of the plane a, is reversed with the plane,

∴ the angle OP makes with OQ, the axis of the plane b, is made greater or less than before by two right angles,

∴ the diedral $a'b = \gamma \pm \pi$, [$a' \equiv$ the reversed plane a.

the diedral $ca' = \beta \pm \pi$, and the parts of the new triangle are

$2\pi - a, \quad b, \quad c, \quad \alpha, \quad \beta \pm \pi, \quad \gamma \pm \pi.$

So, if the plane b be reversed, and the planes c, a not, the parts of the triangle so found (third figure) are

$a, \quad 2\pi - b, \quad c, \quad \alpha \pm \pi, \quad \beta, \quad \gamma \pm \pi.$

So, if the plane c be reversed, and the planes a, b not, the parts of the triangle so found (fourth figure) are

$a, \quad b, \quad 2\pi - c, \quad \alpha \pm \pi, \quad \beta \pm \pi, \quad \gamma.$

So, if the two planes b, c be reversed, and the plane a not, the parts of the triangle so found (fifth figure) are

$a, \quad 2\pi - b, \quad 2\pi - c, \quad \alpha, \quad \beta \pm \pi, \quad \gamma \pm \pi.$

So, if the two planes c, a be reversed, and the plane b not, the parts of the triangle so found (sixth figure) are

$2\pi - a, \quad b, \quad 2\pi - c, \quad \alpha \pm \pi, \quad \beta, \quad \gamma \pm \pi.$

So, if the two planes a, b be reversed, and the plane c not, the parts of the triangle so formed (seventh figure) are

$2\pi - a, \quad 2\pi - b, \quad c, \quad \alpha \pm \pi, \quad \beta \pm \pi, \quad \gamma.$

So, if the three planes a, b, c be all reversed, the parts of the triangle so found (eighth figure) are

$2\pi - a, \quad 2\pi - b, \quad 2\pi - c, \quad \alpha, \quad \beta, \quad \gamma.$

Let the line α have its direction reversed and A′ become the positive end;
then the triedral O-A′BC gives eight triangles, which may be got from those of the triedral O-ABC by noting, that:
with the three planes a, b, c fixed, the diedrals ca, ab, and the face angle BOC are unchanged,
the diedral bc is reversed, being viewed from the other end of α, the co-line of the two planes b, c,
and the face angle COA is replaced by COA′, and AOB by A′OB.

E.g. the parts of the triangle that mates with the ideal triangle are $a,\quad b\pm\pi,\quad c\pm\pi,\quad -\alpha,\quad \beta,\quad \gamma$;
and the other seven may be got in like manner.

So, if the line β be reversed, or the line γ, there are eight new triangles symmetric with the eight triangles of O-A′BC.

If the two lines β, γ be reversed, and α not;
then the diedral bc is unchanged, the diedrals ca, ab are reversed, the face angles COA, AOB are replaced by C′OA′, AOB′, and B′OC′ is equal to BOC.

E.g. the parts of the triangle that mates with the ideal triangle are $a,\quad b\pm\pi,\quad c\pm\pi,\quad \alpha,\quad -\beta,\quad -\gamma$;
and the other seven may be got in like manner.

So, if the two lines γ, α be reversed, or the two lines α, β, there are eight new triangles symmetric with the eight triangles of O-AB′C′.

If the three lines α, β, γ be all reversed;
then, for any one position of the three planes the diedrals are all replaced by their opposites, and the new face angles B′OC′, C′OA′, A′OB′ are equal to BOC, COA, AOB.

E.g. the parts of the triangle that mates with the ideal triangle are $a,\quad b,\quad c,\quad -\alpha,\quad -\beta,\quad -\gamma$.

The parts of thirty-two of these sixty-four triangles are shown in the table below, and the rest may be written by symmetry. In this table the angles are all set down as positive, the negative angles $-\alpha,\quad \alpha-\pi\cdots$ being replaced by their next greater positive congruents.

These triangles have all their parts positive and less than four right angles, and they are called the *primary triangles* of the three co-pointar lines α, β, γ, always naming the lines in this order. A triangle that has parts negative or greater than four right angles may be reduced to one of these by adding or subtracting multiples of four right angles.

A B C	$2\pi-a,\ b,\ c,$ $\alpha,\ \pi+\beta,\ \pi+\gamma.$	$a,\ 2\pi-b,\ 2\pi-c,$ $\alpha,\ \pi+\beta,\ \pi+\gamma.$
$a,\ b,\ c,$ $\alpha,\ \beta,\ \gamma.$	$a,\ 2\pi-b,\ \pi+\gamma,$ $\pi+\alpha,\ \beta,\ \pi+\gamma.$	$2\pi-a,\ b,\ 2\pi-c,$ $\pi+\alpha,\ \beta,\ \pi+\gamma.$
$2\pi-a,\ 2\pi-b,\ 2\pi-c,$ $\alpha,\ \beta,\ \gamma.$	$a,\ b,\ 2\pi-c,$ $\pi+\alpha,\ \pi+\beta,\ \gamma.$	$2\pi-a,\ 2\pi-b,\ c,$ $\pi+\alpha,\ \pi+\beta,\ \gamma.$
A′ B′ C′	$2\pi-a,\ b,\ c,$ $2\pi-\alpha,\ \pi-\beta,\ \pi-\gamma.$	$a,\ 2\pi-b,\ 2\pi-c,$ $2\pi-\alpha,\ \pi-\beta,\ \pi-\gamma.$
$a,\ b,\ c,$ $2\pi-\alpha,\ 2\pi-\beta,\ 2\pi-\gamma.$	$a,\ 2\pi-b,\ c,$ $\pi-\alpha,\ 2\pi-\beta,\ \pi-\gamma.$	$2\pi-a,\ b,\ 2\pi-c,$ $\pi-\alpha,\ 2\pi-\beta,\ \pi-\gamma.$
$2\pi-a,\ 2\pi-b,\ 2\pi-c,$ $2\pi-\alpha,\ 2\pi-\beta,\ 2\pi-\gamma.$	$a,\ b,\ 2\pi-c,$ $\pi-\alpha,\ \pi-\beta,\ 2\pi-\gamma.$	$2\pi-a,\ 2\pi-b,\ c,$ $\pi-\alpha,\ \pi-\beta,\ 2\pi-\gamma.$
A′ B C	$2\pi-a,\ \pi+b,\ \pi+c,$ $2\pi-\alpha,\ \pi+\beta,\ \pi+\gamma.$	$a,\ \pi-b,\ \pi-c,$ $2\pi-\alpha,\ \pi+\beta,\ \pi+\gamma.$
$a,\ \pi+b,\ \pi+c,$ $2\pi-\alpha,\ \beta,\ \gamma.$	$\pi+a,\ 2\pi-b,\ \pi+c,$ $\pi+\alpha,\ 2\pi-\beta,\ \pi+\gamma.$	$\pi-a,\ b,\ \pi+c,$ $\pi+\alpha,\ 2\pi-\beta,\ \pi+\gamma.$
$2\pi-a,\ \pi-b,\ \pi-c,$ $2\pi-\alpha,\ \beta,\ \gamma.$	$\pi+a,\ \pi+b,\ 2\pi-c,$ $\pi+\alpha,\ \pi+\beta,\ 2\pi-\gamma.$	$\pi-a,\ \pi-b,\ c,$ $\pi+\alpha,\ \pi+\beta,\ 2\pi-\gamma.$
A B′ C′	$2\pi-a,\ \pi+b,\ \pi+c,$ $\alpha,\ \pi-\beta,\ \pi-\gamma.$	$a,\ \pi-b,\ \pi-c,$ $\alpha,\ \pi-\beta,\ \pi-\gamma.$
$a,\ \pi+b,\ \pi+c,$ $\alpha,\ 2\pi-\beta,\ 2\pi-\gamma.$	$\pi+a,\ 2\pi-b,\ \pi+c,$ $\pi-\alpha,\ \beta,\ \pi-\gamma.$	$\pi-a,\ b,\ \pi-c,$ $\pi-\alpha,\ \beta,\ \pi-\gamma.$
$2\pi-a,\ \pi-b,\ \pi-c,$ $\alpha,\ 2\pi-\beta,\ 2\pi-\gamma.$	$\pi+a,\ \pi+b,\ 2\pi-c,$ $\pi-\alpha,\ \pi-\beta,\ \gamma.$	$\pi-a,\ \pi-b,\ c,$ $\pi-\alpha,\ \pi-\beta,\ \gamma.$

DETERMINATION OF THE SPECIES OF THE PARTS.

These sixty-four triangles have been divided into two classes called *proper triangles* and *improper triangles*.

To the first class belong the ideal triangle . . . 1
and those got from the ideal triangle by reversing :

one side or one angle, ...	6
one side and the opposite angle, ...	3
two sides and one opposite angle, ...	6
two angles and one opposite side, ...	6
two sides and the two opposite angles, . . .	3
the three sides and two angles, ...	3
the three angles and two sides, ...	3
all the sides and angles, ...	1
in all, thirty-two proper triangles.	32

The other thirty-two triangles are improper triangles; and it will appear that the upper signs of Delambre's formulæ must be used in solving proper triangles, and the lower signs in solving improper triangles.

Certain limitations must also be observed, as below :

Let $a, b, c, \alpha, \beta, \gamma$ be the parts of the ideal triangle,
and $a', b', c', \alpha', \beta', \gamma'$ the parts of any primary triangle;
let $s' \equiv \frac{1}{2}(a'+b'+c')$ and $\sigma' \equiv \frac{1}{2}(\alpha'+\beta'+\gamma')$; then :

1. If the data make a solution possible, the products

$$\sin s' \sin (s'-a') \sin (s'-b') \sin (s'-c'),$$
$$\sin \sigma \sin (\sigma'-\alpha') \sin (\sigma'-\beta') \sin (\sigma'-\gamma'),$$

are both positive, since they are perfect squares. [th. 5, cr. 3.

2. $s' = (s'-a') + (s'-b') + (s'-c')$,
$\sigma' = (\sigma'-\alpha') + \sigma'-\beta') + (\sigma'-\gamma')$.

3. The sixty-four triangles show but ten distinct type-forms, so far as concerns their sides :

$a'=a,$	$a'=a,$	$a'=a,$	$a'=a,$	$a'=a,$
$b'=b,$	$b'=2\pi-b,$	$b'=\pi+b,$	$b'=\pi+b,$	$b'=\pi-b,$
$c'=c;$	$c'=2\pi-c;$	$c'=\pi+c;$	$c'=\pi-c;$	$c'=\pi-c;$

$$\begin{array}{lllll} a'=2\pi-a, & a'=2\pi-a, & a'=2\pi-a, & a'=2\pi-a, & a'=2\pi-a, \\ b'=b, & b'=2\pi-b, & b'=\pi+b, & b'=\pi+b, & b'=\pi-b, \\ c'=c; & c'=2\pi-c; & c'=\pi+c; & c'=\pi-c; & c'=\pi-c; \end{array}$$

and there are eight possible groups of consistent inequalities:

$$\begin{array}{llll} 0<s'<\pi, & 0<s'-a'<\pi, & 0<s'-b'<\pi, & 0<s'-c'<\pi; \\ 2\pi<s'<3\pi, & 0<s'-a'<\pi, & 0<s'-b'<\pi, & 0<s'-c'<\pi. \end{array}$$

$$\begin{array}{llll} \pi<s'<2\pi, & \pi<s'-a'<2\pi, & 0<s'-b'<\pi, & 0<s'-c'<\pi; \\ \pi<s'<2\pi, & -\pi<s'-a'<0, & 0<s'-b'<\pi, & 0<s'-c'<\pi. \end{array}$$

$$\begin{array}{llll} \pi<s'<2\pi, & 0<s'-a'<\pi, & \pi<s'-b'<2\pi, & 0<s'-c'<\pi; \\ \pi<s'<2\pi, & 0<s'-a'<\pi, & -\pi<s'-b'<0, & 0<s'-c'<\pi. \end{array}$$

$$\begin{array}{llll} \pi<s'<2\pi, & 0<s'-a'<\pi, & 0<s'-b'<\pi, & \pi<s'-c'<2\pi; \\ \pi<s'<2\pi, & 0<s'-a'<\pi, & 0<s'-b'<\pi, & -\pi<s'-c'<0. \end{array}$$

with like type-forms and like groups of the angles.

These inequalities are relied on to determine the species of the parts sought. There are always two solutions: real and separate, real and coincident, or imaginary.

E.g. in case (a), given b, c, α: the angles $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\beta-\gamma)$ are both two valued.

But $0<\frac{1}{2}(\beta+\gamma)<2\pi$, $-\pi<\frac{1}{2}(\beta-\gamma)<\pi$; and together they give only two values each to β, γ between 0 and 2π;

and a is found from the values of β, γ without ambiguity.

So, in (c) γ has two values, and a, α have single values for each value of γ.

So, (b, d) follow (a, c), and (e, f) show no ambiguity.

GRAPHICAL SOLUTION.

The graphical solution of a primary triangle is made in the same way as that of the ideal triangle, angles greater than two right angles being replaced by their next less negative congruents. If any construction be possible, there are two constructions, which may be separate or coincident.

§ 19. SPHERICAL ASTRONOMY.

THE CELESTIAL SPHERE.

In astronomy the elements of position of a heavenly body are distance and direction ; in spherical astronomy only one of these elements, direction, is regarded, and that is usually referred to the earth's centre. For this purpose all stars may be considered as at the same distance from the earth's centre upon the surface of a sphere of arbitrary radius called the *celestial sphere.*

The trace of the plane of the earth's equator on this sphere is the *celestial equator,* whose poles (north and south) are the traces of the earth's axis.

The *ecliptic* is a great circle of the celestial sphere, the sun's apparent path in one year due to the motion of the earth around the sun ; it cuts the equator in two points, the *vernal* and the *autumnal equinox,* which are passed through by the sun, about March 20 and September 23. The *obliquity* of the ecliptic is the nearly constant angle of 23° 27′ between the planes of the ecliptic and equator.

Secondaries to any great circle, or *primary,* are great circles cutting it and therefore its parallels at right angles. Secondaries to the celestial equator are *hour-circles* or meridians.

To any observer the *sensible horizon* is a plane touching the earth's surface at the point of observation ; and a plane parallel to this plane through the earth's centre traces out on the celestial sphere the *rational horizon,* whose poles, *zenith* and *nadir,* are the traces of a vertical line, and whose secondaries are *vertical circles.* One of the vertical circles is also an hour-circle, the observer's *celestial meridian,* and passes through his zenith and nadir, and the north and south poles of the celestial sphere ; its plane is the same with that of the observer's terrestrial meridian, and it meets the plane of his sensible horizon in his *meridian line.* The vertical circle that is perpendicular to the meridian is the observer's *prime vertical,* and it goes through the east and west points of his horizon.

SPHERICAL COORDINATES.

As the position of a point on the earth's surface is defined by means of two coordinates (latitude and longitude), the standards of reference being a convenient great circle (the equator) and a convenient point on it (the point where it is crossed by the meridian of Greenwich); so, the position of a star at any instant on the celestial sphere may be defined in either of three ways:

1. *As to the celestial equator:*

The *declination* of a star is its angular distance (north or south) from the celestial equator measured upon its hour-circle; and the arc of the equator intercepted between this circle and the vernal equinox is the star's *right ascension;* it is reckoned eastward from the vernal equinox from 0° to 360°. The complement of the declination is the polar distance.

Instead of a star's right ascension its *hour-angle* is often used, in problems that involve diurnal motion, to define its hour-circle at any instant; this angle is the angle at the pole between the observer's celestial meridian and the star's hour-circle, and is counted from the meridian, positive towards the west and negative towards the east. The right ascension of a fixed star changes very little, since the vernal equinox is nearly fixed on the celestial sphere; the hour-angle changes every moment.

2. *As to the ecliptic:*

The *latitude* of a star is its angular distance from the ecliptic measured on a secondary; and the arc of the ecliptic intercepted between the vernal equinox and this secondary, measured eastward, is the star's *longitude.*

3. *As to the horizon:*

The *altitude* of a star is its angular distance from the horizon measured on a vertical circle; and the arc of the horizon intercepted between this circle and the south point of the horizon is the star's *azimuth.* Owing to the rotation of the celestial sphere, the horizon-coordinates change every moment.

RELATIONS BETWEEN ECLIPTIC-COORDINATES AND EQUATOR-COORDINATES.

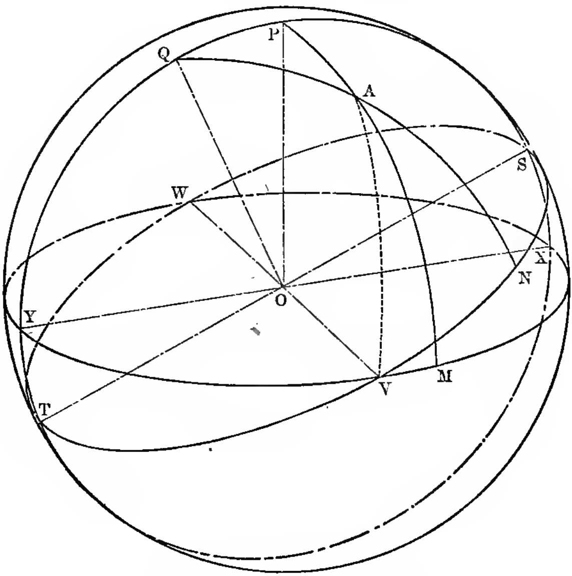
On the celestial sphere let P be the pole of the equator, Q the pole of the ecliptic;

then the great circle through P, Q is the common secondary of the equator and the ecliptic.

Let V, W be the vernal and the autumnal equinox at quadrantal distances from S, T;

let PAM be the hour-circle of a star A, and QAN the secondary to the ecliptic;

then VM, MA are the right ascension and declination of A, and VN, NA are its longitude and latitude.



These four coordinates of any fixed star are subject to only slight variations in any one year; they are recorded for the principal stars in a yearly almanac, with the data for computing

the variations; the sun's declination is recorded for each day or half day, and may be got for any hour and minute by interpolation.

The spherical angle XVS is the obliquity of the ecliptic, and, since VX, VS are quadrants, XVS is measured by the arc XS; arcs XS, PQ, YT are each 23° 27′, and arcs SP, YQ are each 66° 33′.

Equator-coordinates may be converted into ecliptic-coordinates:

when VM, MA are given in the right spherical triangle MVA, the arc VA and the angle MVA may be found;

the angle NVA is found by subtracting the obliquity, and the triangle NVA may be solved for VN, NA; so conversely.

THE SUN'S ANNUAL MOTION.

The particular case of the sun is simpler: since his apparent annual path is the ecliptic, his latitude is always zero, and his right ascension, declination, and longitude are the arc-abscissa, arc-ordinate, and arc-distance of a given angle, the obliquity; his declination increases from 0° at V on March 21 to 23° 27′ at S on June 21 (the summer solstice), then decreases to 0° at W on September 21, and to ⁻23° 27′ at T on December 22 (the winter solstice), then increases to 0° at V; his right ascension and longitude are equal at 0°, 90°, 180°, 270°, 360°.

QUESTIONS.

1. The altitude of a circumpolar star at upper *transit* across meridian is 60°, and at lower transit 40°: find the declination of the star.

2. The vernal equinox culminated (reached its highest point) at $0^h 10^m 13^s$, and a certain star culminated at $2^h 5^m 10^s$: find its right ascension.

3. Find the latitude and longitude of a star whose right ascension is $5^h 13^m$, and declination 60°.

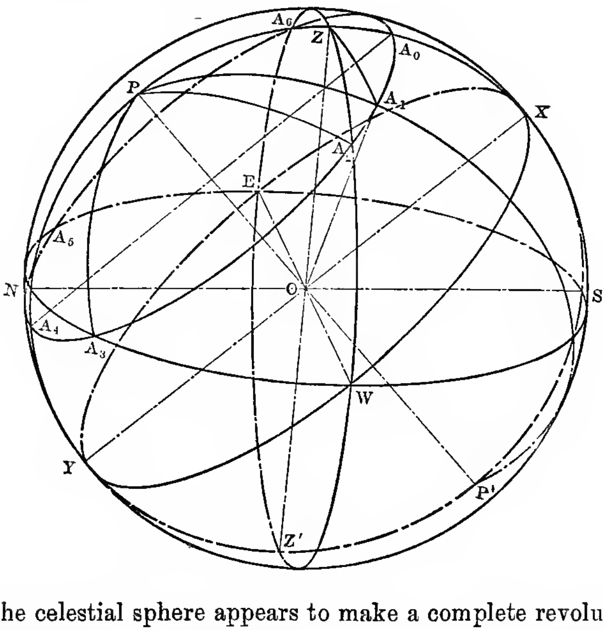
4. When the sun's declination is 15°, find his right ascension and longitude.

RELATIONS BETWEEN EQUATOR-COORDINATES AND HORIZON-COORDINATES.—THE ASTRONOMICAL TRIANGLE.

On the celestial sphere let P be the pole of the equator XY, and Z that of the horizon NS;

then the great circle through PZ is the celestial meridian, the common secondary of equator and horizon;

let ZWZ′E be the prime vertical perpendicular to both meridian and horizon and meeting both equator and horizon in the east and west points.



The celestial sphere appears to make a complete revolution on its axis PP′ in about $23^h\ 56^m\ 4^s$ of civil time. This is the interval between two successive transits of any fixed star, and is a *sidereal day.* A sidereal clock shows 0 hours when the vernal equinox culminates; and the hours are marked from 0 to 24. The sidereal time of a star's transit gives its exact

right ascension, which may be converted into angular measure at the rate of 15° to a sidereal hour, or 1° to 4 minutes, 15′ to 1 minute, 1′ to 4 seconds, and so on.

The hour-circle of the star A coincides with the meridian in the position PA_0 bearing due south as seen from O, and the star has then its greatest altitude:

in the position PA_2 the star is on the prime vertical and bears due west;

in the position PA_3 the star sets below the horizon;

it reaches its greatest depression at A_4 when its hour-circle passes over the meridian bearing due north;

it rises at A_5, reaches the prime vertical at A_6 bearing due east, and culminates again at A_0.

The spherical triangle ZPA_1 for any position of the star A is the *astronomical triangle:*

its sides ZA_1, PA_1 are the co-altitude and co-declination of A_1;

the angles ZPA_1, PZA_1 are the supplement of the hour-angle and of the azimuth of A;

and the side PZ is the observer's co-latitude.

For ∵ this co-latitude is the angle between the earth's axis and the vertical line at the point of observation,

and the traces of these lines on the celestial sphere are P, Z,

∴ the arc PZ measures the observer's co-latitude.

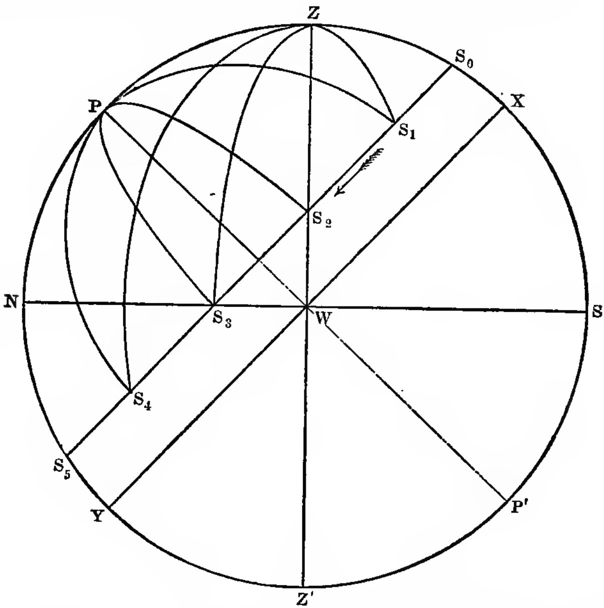
When the latitude is known the relations between the sides and angles of this triangle give the relations between the star's equator- and horizon- coordinates.

The observer's latitude may be determined, once for all, by the astronomical triangle when the declination, the altitude, and either the azimuth or the hour-angle of a heavenly body are known for some instant. If at the time of observation the body be on the meridian, the hour-angle is zero, and the azimuth either zero or 180°; if it be on the prime vertical, the azimuth is ±90°; if it be on the horizon, the altitude is zero; and in all these cases the computation of latitude is simplified.

THE SUN'S DIURNAL MOTION.—SOLAR TIME.

The sun's hour-circle PS_0 coincides with the observer's meridian at noon; the hour-angle ZPS_1 at any instant measures the time of observation from noon; the angle ZPS_3' measures the time of sunset.

The greater the declination XS_0, the greater is the hour-angle of setting, ZPS_3, the longer the day, and the shorter the night. The day is longest in the northern hemisphere when the declination is greatest, June 21, the summer solstice.



When the declination is zero, the diurnal path S_0S_5 coincides with the equator and is bisected by the horizon; the day is then equal in duration to the night, and hence the term equinox. When the declination is 23° 27′ s., the day is shortest in north latitudes, and the night longest (winter solstice).

The interval between two successive transits of the sun over

the same meridian is an *apparent solar day.* This interval varies, from two causes: the obliquity of the ecliptic, and the variability of the sun's apparent motion in the ecliptic.

The *mean sun* is an imaginary body, supposed to move uniformly in the equator with the annual period, and with the average velocity, of the true sun. It culminates at *civil* or *mean noon*, and the constant interval between two successive transits is a *mean solar day.* This interval is divided into hours, minutes, and seconds. A second of mean solar time is the ordinary time-unit, and is the same fraction of a mean solar day as a sidereal second is of a sidereal day. The *mean solar time* is the hour-angle of the mean sun, at any instant; the *apparent solar time* is the hour-angle of the true sun. The angle between the mean and true hour-circles is recorded for each day, in the almanac, as the *equation of time.* It varies throughout the year between 0 and about ± 16 minutes of time.

The *astronomical day* begins at mean noon, and the hours are numbered from 0 to 24.

In what follows, apparent time is used.

QUESTIONS.

1. The meridian altitude of the sun's centre was $25° 38' 30''$ S., and his declination $22° 18' 14''$ S.: find the latitude.

2. The meridian altitude of Jupiter was $50° 20' 8''$ S., and his declination $18° 47' 37''$ N.: find the observer's latitude.

3. The sun crossed the prime vertical at an altitude of $54°$: find the observer's latitude and the time of day, the sun's declination, got by interpolation for the approximate time of day, being $18° 30'$.

Here, $\text{ZS}_2 = 36°$, $\text{PS}_2 = 71° 30'$, $\text{PZS}_2 = 90°$: find ZP, ZPS_2.

4. Find the observer's latitude in ex. 1, page 176.

In what latitude will this star just graze the horizon?

5. If the sun's declination be $22° 26'$ N., and altitude $40° 55'$ at 3 P.M., find the observer's latitude.

In this example, $\text{ZS}_1 = 49° 5'$, $\text{PS}_1 = 67° 34'$, $\text{ZPS}_1 = 3\text{ h.} = 45°$, and the co-latitude, PZ, is to be found.

6. In latitude 13° 17′ N. the sun's altitude was 36° 37′, his declination was 22° 10′ S.: find his hour-angle.

7. If the sun's declination be 17° N., find the time in the afternoon when he will be due west from a place in latitude 51° N.; and find how far from the west point he will set (his *amplitude* at setting).

8. If the sun be due west at setting (amplitude zero), find his declination and the time of year.

9. If the time of sunset be sought on any given day, a quadrantal triangle PZS_3 may be solved for the hour-angle ZPS_3.

If the sun's declination be 14° S. and the latitude 42° N., find the time and amplitude of sunrise and sunset.

10. Find the time of setting in ex. 8.

11. Find the length of the longest day in Ithaca, excluding twilight, latitude 42° 30′ N.

12. Find the lowest north latitude in which the sun does not set on the longest day, nor rise on the shortest day.

13. Find the time of sunrise in Boston, latitude 42° 21′ N., on the shortest day of the year, and the sun's amplitude.

14. The phenomenon of twilight is due to the reflection and refraction of some of the sun's rays toward the observer's eye when the direct rays are intercepted: it begins or ends when the sun is about 18° below the horizon.

How long does twilight last in Boston on the shortest day?

Given $ZS_4 = 90° + 18°$, PS_4, ZP: find ZPS_4, and subtract the hour-angle of sunset, ZPS_3.

15. Find the length of the longest day in Ithaca, including morning and evening twilight.

16. In what latitude does the sun get just 18° below the horizon on the longest day, so that twilight lasts all night?

Here, $NS_5 = 18°$, $PS_5 = 66° 33'$: find the co-latitude ZP.

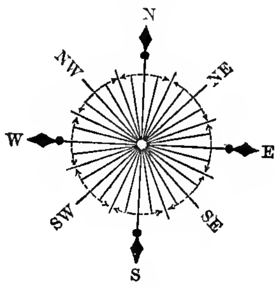
17. Given the declination of Aldebaran, 16° 17′ N.: find his altitude and azimuth to an observer at Boston when the hour-angle of this star is $3^h 25^m 12^s$; and find the hour-angle and amplitude at rising and setting.

§ 20. NAVIGATION.

When a mariner cannot make celestial observations, he has recourse to *dead-reckoning; i.e.* he computes the position of his ship from the latitude and longitude of her starting-point or of the place of last observation, and the records of sailing. This dead-reckoning is the subject of navigation proper, as distinguished from nautical astronomy.

The rate of sailing is usually recorded every hour, and is measured by the *log-line.* This is a line wound on a reel and attached to a small quadrantal piece of board. The quadrant is loaded on the arc with lead to keep it upright when thrown into the water and prevent its moving forward toward the ship while the line is running out. The log-line is divided into *knots,* each a hundred-twentieth part of a nautical mile, so that the number of knots run out in half a minute gives the ship's hourly rate in miles.

Bearings at sea are given in *points* and *quarter-points,* counted from each of the eight cardinal points, two points each way.



The direction of sailing at any time is shown by the *mariner's compass.*

The reading of the compass is to be corrected for variation, deviation, and leeway.

The *variation* is the angle between the magnetic and true meridians; it is found, for various places, by astronomical observations, and laid down on the nautical charts.

The *deviation* is the angle of deflection of the needle from the magnetic meridian, caused by the iron of the ship; it is found, for a given ship and a given direction, by special experiments.

When there is a side wind, the angle which the ship's track makes with her fore-and-aft line is the *leeway:* it is found, for a given ship, a given freight, and a given obliquity and velocity of the wind, by special experiments.

The corrected reading is the *course;* it is the angle between the ship's true meridian and her true direction of motion. In what follows, the corrections are supposed to have been made, so that the given courses are the true courses. When the course is kept constant, the ship's track crosses every meridian at the same angle; the path is neither straight nor circular, but a spiral, the *loxodrome* or *rhumb-line*, that goes round and round the earth's surface, coming nearer and nearer to the pole; and its length is the *distance.*

The meridian length between the first and last parallel of latitude is the *difference of latitude* made by the ship.

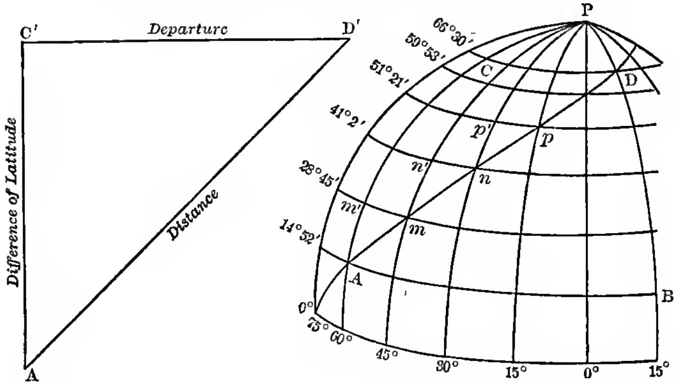
The *departure* is her easting or westing from her first meridian; it is measured as follows: if she sail on a parallel of latitude, the departure is the distance made on the parallel; if she sail on a loxodrome, the departure for each successive instant is measured on the parallel she is then crossing, and the limit of the sum of these infinitesimal departures is the total departure.

The unit of length is the *nautical mile,* about 6080 feet, a sixtieth part of a degree of a great circle of the earth. Sixty nautical miles are a little more than sixty-nine statute miles.

In what follows the earth is regarded as a perfect sphere. The error thus introduced is too small to be taken into account in any calculations whose data are derived from the log-line and compass.

PLANE SAILING.—RELATIONS BETWEEN COURSE, DISTANCE, DIFFERENCE OF LATITUDE, AND DEPARTURE.

Let AD be the rhumb-line, PA, PD the first and last meridians,
Pm, Pn $\cdots$ meridians at equal small intervals;



let $m'm$, $n'n$, $\cdots$ be the small arcs intercepted on successive parallels;
then the total departure from A to D is the limit of the sum $m'm + n'n + \cdots$, when the meridians are taken very close together. [df. dep.

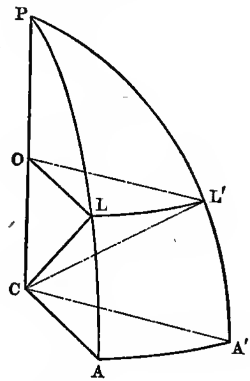
The infinitesimal triangles Amm', mnn' may be treated as right plane triangles; and since the course is constant they are similar. The elements of the motion are thus given by a series of infinitesimal right plane triangles, the sum of whose hypotenuses is the distance, of whose bases is the departure, and of whose altitudes is the difference of latitude. These three sums, and the course, have the same relations to each other as the parts of any one of the elemental triangles; hence they may be accurately represented by the parts of a right plane triangle. For this reason, although the sphericity of the earth is taken into account, the term *plane sailing* may be applied to any problem into which the difference of longitude does not enter; and the solution is effected by the rules for the solution of right plane triangles.

PARALLEL SAILING.—RELATIONS BETWEEN A DISTANCE SAILED ON A GIVEN PARALLEL OF LATITUDE AND THE DIFFERENCE OF LONGITUDE.

THEOR. 23. *The length of an arc of a parallel of latitude is the product of the length of the equatorial arc of the same number of degrees by the cosine of the latitude of the parallel.*

For let P be a pole of the earth, C is centre, A, A′ any two points on the equator, PA, PA′ two meridians cutting a parallel of latitude in L, L′;

let O be the centre of the arc LL′;



then ∵ arc LL′ : arc AA′ = OL : CA [geom.

= OL/CL = cos ACL = the cosine of the latitude ;

∴ LL′ = AA′ · the cosine of the latitude. Q.E.D.

MIDDLE LATITUDE SAILING.—APPROXIMATE RELATION BETWEEN THE DIFFERENCE OF LONGITUDE AND THE DEPARTURE ON A LOXODROME.

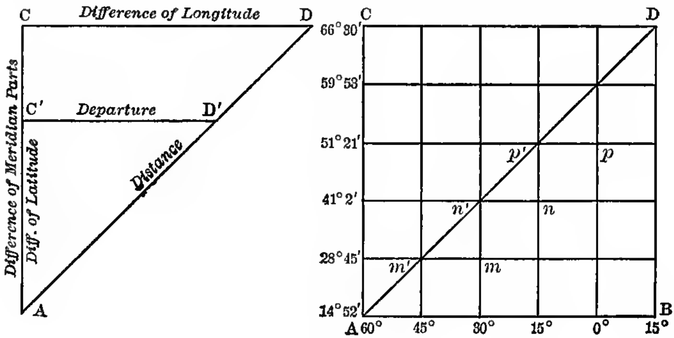
The departure from A to D lies, in value, between AB and CD, and for short distances is nearly the same as the ship makes if she sail between the same two meridians on the mid-parallel ; *i.e.* the parallel whose latitude is half the sum of the latitudes of A and D. Hence the departure from A to D is taken equal

to the product of the difference of longitude of A and D by the cosine of their middle latitude. [theor. 23.

The difference of longitude is thus connected with the other elements of the ship's path.

MERCATOR'S PROJECTION.—ACCURATE RELATION BETWEEN THE DIFFERENCE OF LONGITUDE AND THE DEPARTURE ON A LOXODROME.

Project the figure of page 184 on a plane surface as follows:



1. Draw a horizontal line for the equator, and vertical lines at equal intervals for the meridians.

It follows that the projection $m'm$ of any arc of a parallel is equal to the corresponding arc of the equator, and is therefore multiplied by a projecting factor, the secant of its own latitude.

2. Draw a straight line cutting the meridians at the constant angle given by the course.

It follows that the angles of each small plane triangle remain the same; so that while each triangle is enlarged, its shape is preserved, and $m'n'$ or mn' has the same projecting factor as mm', and $n'p'$ or np' the same projecting factor as $n'n$, and so on.

Each small portion of the meridian in the neighborhood of any parallel is therefore multiplied by the secant of the latitude of that parallel, and the total length of the projection of

any given portion of a meridian is the limit of the sum of these products, when the parts are taken indefinitely small. In practice it is sufficiently accurate to take each part as two minutes or nautical miles, and to use as its projecting factor the secant of the latitude of its middle point.

E.g. the meridian-arc between the equator and latitude 13° 16′ projects into a distance on the chart equal to the sum

$$2\,(\sec 1' + \sec 3' + \sec 5' + \cdots + \sec 795')$$

in nautical miles, on the assumed scale.

This distance is computed and tabulated as the *meridional part* for 13° 16′. In computing such a table, each entry may be used in succession to find the next one, *e.g.* the meridional part for 36′ is found from that for 34′ by adding $2 \sec 35'$.

The difference between the meridional parts for two latitudes is their *meridional difference* of latitude.

In the figure above, AC is the meridional difference of latitude from A to D, and CD is the difference of longitude;

and dif. long./merid. dif. lat. = tan course

= dep./true dif. lat. [plane sailing.

These equations connect the difference of longitude with the other elements of the ship's motion:

E.g. given the latitude and longitude of A, the course and distance from A to D:

find by plane sailing the departure, the difference of latitude, and the latitude of D;

find the meridional difference of latitude by subtracting meridional part for latitude A from that for latitude D;

compute the difference of longitude from the above relations.

NOTE. The student of the calculus will see that the exact meridional part for latitude λ is

$$\int_0^\lambda \sec\lambda \cdot d\lambda = \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right), \text{ in radians};$$

and this result may be reduced to nautical miles, as follows:

$$\because \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right) = \log_{10} \tan\left(45° + \tfrac{1}{2}\lambda\right) \cdot 2.3026,$$

and $1^r = 3437.75'$, $2.3026 \cdot 3437.75 = 7916$,

$$\therefore \log_e \tan\left(45° + \tfrac{1}{2}\lambda\right) = \log_{10} \tan\left(45° + \tfrac{1}{2}\lambda\right) \cdot 7916.$$

E.g. if $\lambda = 13° 15'$,

then $45° + \frac{1}{2}\lambda = 51° 37' 30''$,

the merid. part $= 0.10134 \cdot 7916 = 802$ nautical miles,

and this latitude is enlarged in the ratio $802 : 795$.

TRAVERSE SAILING.

PROB. 5. TO REDUCE THE RESULT OF SEVERAL SUCCESSIVE COURSES AND DISTANCES TO A SINGLE COURSE AND DISTANCE.

(*a*) *The latitude of the starting-point not given:*

Compute each separate difference of latitude and departure by plane sailing;

take the algebraic sum of the separate differences of latitude for the value of the direct difference of latitude,

and the algebraic sum of the departures for an approximate value of the direct departure;

find the direct course and distance by plane sailing.

(*b*) *The latitude of the starting-point given:*

Compute the separate differences of latitude by Mercator's or middle-latitude sailing;

take their algebraic sum for the direct difference of latitude, and so for the differences of longitude;

from these find the direct departure by Mercator's or middle-latitude sailing;

find the direct course and distance by plane sailing.

NOTE. The first course and distance entered are usually got by *taking a departure, i.e.* by taking the bearing and distance of some object of known latitude and longitude; the reverse of these are entered on the log-slate as the first course and distance.

GREAT CIRCLE SAILING.

The shortest distance between two places is the great circle arc joining them; it does not cut all the meridians at the same angle; hence to keep on a great circle the ship must contin-

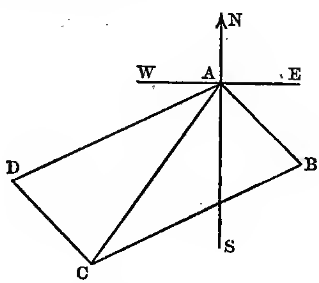
ually change her course. By means of a chart several places on the great circle may be determined, and if the ship lay her course for these on successive rhumb-lines, her path will differ little from the circular arc.

The *elements* of the great circle track between two given places are the distance, the first and last courses, and the highest latitude passed through. These are got from the spherical triangle whose vertical angle at the pole is the difference of longitude of the two places, and whose sides are their co-latitudes.

CURRENTS.

In order to ascertain the *set* and *drift* of a current, *i.e.* its direction and velocity, a boat is taken a short distance from the ship and kept stationary by letting down a heavy weight; the log is thrown from the boat, and the direction in which it is carried, *i.e.* the set of the current, is taken by the boat compass, while the drift is given by the number of knots run off in half a minute. The effect of the current is considered equivalent to an independent course.

E.g. if a ship sail 10 knots an hour in a current setting S.E. 5 miles an hour, what course must she lay to make a place whose bearing is S.W. by S.?



(*a*) *By construction and measurement.*

Take AB pointing S.E., and equal to 5 on any scale;

take AC pointing S.W. by S.;

with B as centre, and radius BC equal to 10, cut AC in the point C; complete the parallelogram ABCD:
the angle SAD is the course sought.

(*b*) *By computation.*

In the triangle ABC, the sides AB, BC and the angle BAC being known, compute the angle BCA, and thence the course SAD.

TACKING.

A ship is on the *starboard tack* when the wind is on her right, on the *port tack* when the wind is on her left; she is *close-hauled* on either tack when she sails as nearly as possible toward the point whence the wind blows.

If when close-hauled she find her destination lying between her path and the wind, then she cannot reach it on this single tack; but she may continue till the angle that the direction of her destination makes with the wind is just equal to her angle of close-haul, and then run in close-hauled on the other tack.

E.g. if a ship can sail within 6 points of the wind on the port tack, and within $5\frac{1}{2}$ points on the starboard tack,
find her course and distance on each tack to reach, in the shortest time, a point 15 miles N.W., with the wind due west:

(*a*) *By construction and measurement.*

Take AC pointing N.W., and equal to 15 on any scale;
for the port tack draw AB 6 points to the right of the wind;
for the starboard tack draw AD $5\frac{1}{2}$ points to the left of the wind;
from the point C draw CB parallel to DA;
measure AB and BC for the distances on each tack.

(*b*) *By computation.*

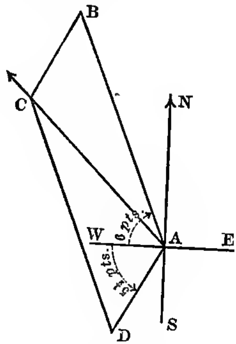
In the triangle ABC, $AC = 15$, $A = 6 - 4 = 2$ points,

$$B = \overline{8-6} + \overline{8-5\tfrac{1}{2}} = 4\tfrac{1}{2} \text{ points}, \quad C = 5\tfrac{1}{2} + 4 = 9\tfrac{1}{2} \text{ points},$$

check: $A + B + C = 16$ points; compute AB, BC.

The answer is the same whichever tack be taken first.

NOTE. The surface of the earth is supposed to be flat within the limits of these problems. They come usually under case (*b*) in compound-course sailing.



QUESTIONS.

1. A ship sails due west 117 miles from a point in lat. 38° N., long. 16° E.: find the longitude reached. [13° 31′ 30″ E.

2. In what latitude is a degree of longitude half as long as at the equator?

3. Sail S. E. 67 miles from New York light, lat. 40° 28′ N., long. 74° 8′ W.: by middle-latitude sailing find the latitude and longitude of the point reached. [39° 40′ 36″ N., 73° 6′ 6″ W.

4. Find the course and distance from Montauk Point, 41° 4′ N., 72° W., to Martha's Vineyard, 41° 17′ N., 70° 48′ W.

[N. 76° 30′ 40″ E., 55.73 miles by middle-latitude sailing;
N. 76° 43′ E., 56.58 miles by Mercator's sailing.

5. A ship sails from a point 14° 45′ N., 17° 33′ W., on a course S. 28° 7′ 30″ W., till she reaches longitude 29° 26′ W.: find by Mercator the distance sailed and the latitude.

[1500 miles, 7° 18′ S.

6. From a point in latitude 50° 10′ S. a ship sails S. 67° 30′ E. till her departure is 957 miles: find by Mercator the distance sailed, the difference of latitude, and the difference of longitude. [1036, 6° 36′ 24″, 26° 53′.

7. A ship starting from a point in latitude 32° N. sails N. 25° E. 16 miles, thence S. 54° E. 11 miles, thence N. 13° W. 7 miles, thence N. 61° E. 5 miles, thence N. 38° W. 18 miles: find the single course and distance that would bring her to the same destination. [(*a*) N. 13° 12′ 20″ E., 32.30 miles; (*b*) N. 11° 40′ 37″ E., 32.12 miles.

8. Find the elements of the great circle track between New York light and Cape Clear, 51° 26′ N., 9° 29′ W.

9. So, between San Francisco, 37° 48′ N., 122° 25′ W., and Cape of Good Hope, 33° 56′ S., 18° 29′ E.